Rewriting (D)Q2-CNF with Arbitrary Free Literals into ∃2-HORN

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12.7.2010

Outline

- Introduction
  - The Class Q2-CNF^b
  - Application: CNF Transformation
  - Q2-CNF^b versus ∃2-HORN^b
- Transforming (D)Q2-CNF^b to ∃2-CNF^b
- Transforming ∃2-CNF^b to ∃2-HORN^b
- Conclusion

Introduction

Free Variables 1/2

We consider QBF with free variables (not bound by a quantifier).

Example: $\forall x\, \exists y\, (\neg x \vee y) \wedge (x \vee \neg y) \wedge (z \vee \neg y)$

Observation:

- closed QBF: either true or false
- QBF with free variables: truth depends on the values of the free variables

Free Variables 2/2

Why do we need free variables?

Given: propositional formula $\phi(z_1,\ldots,z_r)$

Wanted:

shorter QBF $\Phi(z_1,\ldots,z_r)$ with free variables $z_1,\ldots,z_r$

such that $\mathfrak{I}_\Phi(z_1,\ldots,z_r) = \mathfrak{I}_\phi(z_1,\ldots,z_r)$ („equivalence“)

Example:

$$(A \vee \neg B \vee C \vee D) \wedge (A \vee \neg B \vee C \vee \neg E) \wedge (A \vee \neg B \vee C \vee F)$$

Free Variables 2/2

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Given: propositional formula $\phi(z_1,\ldots,z_r)$

Wanted:

shorter QBF $\Phi(z_1,\ldots,z_r)$ with free variables $z_1,\ldots,z_r$

such that $\Im_\Phi(z_1,\ldots,z_r) = \Im_\phi(z_1,\ldots,z_r)$ („equivalence“)

Example:

$$(A \lor \neg B \lor C \lor D) \land (A \lor \neg B \lor C \lor \neg E) \land (A \lor \neg B \lor C \lor F)$$

$$\approx \exists y\, (y \rightarrow (A \lor \neg B \lor C)) \land (y \lor D) \land (y \lor \neg E) \land (y \lor F)$$

The Formula Class Q2-CNF^b

We are interested in formulas of the form

$$\Phi(\mathbf{z}) = Q_1 v_1 ... Q_n v_n \bigwedge_i (\phi_i^b \vee \phi_i^f)$$

where ϕ_i^b is a 2-CNF clause over bound variables

and ϕ_i^f an arbitrary clause over free variables.

Example:

$$\forall x \exists y \, (x \vee y \vee z_1) \wedge (y \vee z_2 \vee z_3 \vee \neg z_4) \wedge (\neg x \vee \neg y \vee z_5) \wedge (\neg y \vee z_6)$$

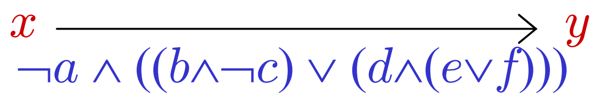
$$\approx (z_1 \vee z_6) \wedge (z_2 \vee z_3 \vee \neg z_4 \vee z_5) \wedge (z_2 \vee z_3 \vee \neg z_4 \vee z_6)$$

Q2-CNF^b is exponentially more powerful than CNF: every propositional formula has a poly-size Q2-CNF^b equivalent (CNF only achieves satisfiability equivalence).

CNF Transformation

Idea: Transformation by Series-Parallel Graphs

Example: $\psi := \neg a \wedge ((b \wedge \neg c) \vee (d \wedge (e \vee f)))$



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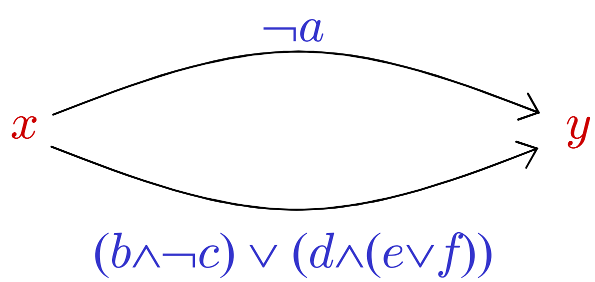

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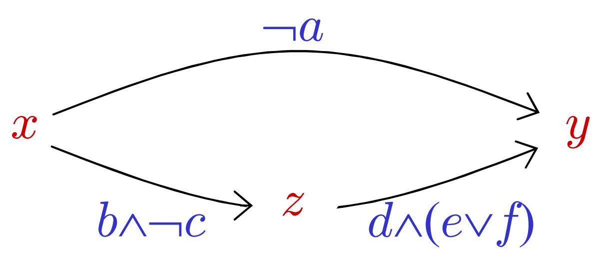

CNF Transformation

Idea: Transformation by Series-Parallel Graphs

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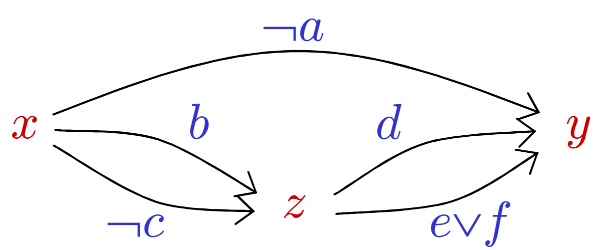


Now map edges $\alpha \xrightarrow[\gamma]{} \beta$ onto clauses $(\neg\alpha \vee \beta \vee \gamma)$.

$$\exists x \exists y \exists z \;\; (\neg x \vee y \vee \neg a) \wedge (\neg x \vee z \vee b) \wedge (\neg x \vee z \vee \neg c)$$

$$\wedge \;\; (\neg z \vee y \vee d) \wedge (\neg z \vee y \vee e \vee f) \wedge x \wedge \neg y$$

$$\approx \exists z \; \neg a \wedge (z \vee b) \wedge (z \vee \neg c) \wedge (\neg z \vee d) \wedge (\neg z \vee e \vee f)$$

Q2-CNF^b versus ∃2-HORN^b 1/2

$$\exists x \exists y \exists z \;\; (\neg x \vee y \vee \neg a) \wedge (\neg x \vee z \vee b) \wedge (\neg x \vee z \vee \neg c)$$

$$\wedge \;\; (\underbrace{\neg z \vee y}_{\text{at most 2 bound literals per clause}} \vee d) \wedge (\neg z \vee y \vee e \vee f) \wedge x \wedge \neg y$$

Actually: ∃2-HORN^b (⊆Q2-CNF^b)

∃(k-)HORN^b:

$\exists v_1 \ldots \exists v_n \bigwedge_i (\phi^b_i \vee \phi^f_i)$

ϕ^b_i a (k-)HORN clause over bound literals

ϕ^f_i an arbitrary clause over free literals

Q2-CNF^b versus ∃2-HORN^b 2/2

So we know: CNF $\subset_{\text{poly-length}}$ ∃2-HORN^b

Main Question: is Q2-CNF^b even more powerful?

2 Aspects to consider:

- Universal and existential quantifiers vs only existential
- 2-CNF vs 2-HORN

Transforming (D)Q2-CNF^b to ∃2-CNF^b

Power of Universal Quantification

Horn formulas cannot benefit much from universal quantification:

$\text{QHORN}^b =_{\text{poly-time}} \exists\text{HORN}^b$ [Bubeck/KB/Zhao '05/'09]

Even for generally more powerful dependency quantification:

$\text{DQHORN}^b =_{\text{poly-time}} \exists\text{HORN}^b$

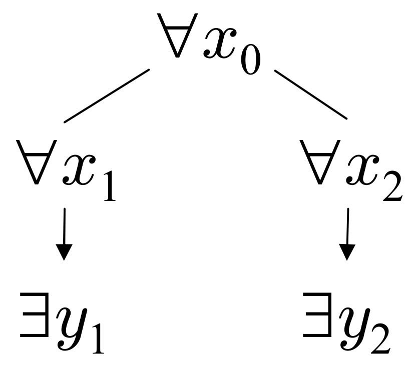
by a quadratic-time transformation. [Bubeck/KB '06]

We are going to see: Situation is similar for 2-CNF.

Dependency Quantifiers 1/2

Consider a quantified formula where:

- y_1 depends on x_0 and x_1
- y_2 depends on x_0 and x_2



No suitable prenex QBF encoding:

$\forall x_0 \forall x_1 \exists y_1 \forall x_2 \exists y_2 \, \phi(x_0, x_1, x_2, y_1, y_2)$

now: y_2 depends on x_0, x_1, x_2

Non-prenex QBF only possible if y_1 and y_2 not used together:

$\forall x_0 \, (\forall x_1 \exists y_1 \alpha(x_0, x_1, y_1)) \wedge (\forall x_2 \exists y_2 \beta(x_0, x_2, y_2))$

Solution: Dependency Quantified Boolean Formulas

$\forall x_0, x_1, x_2 \, \exists y_1(x_0, x_1) \, \exists y_2(x_0, x_2) \, \alpha(x_0, x_1, y_1) \wedge \beta(x_0, x_2, y_2) \wedge \gamma(x, y_1, y_2)$

also possible

Dependency Quantifiers 2/2

Dependency Quantification is generally very powerful:

- QBF: two-player game, 1 univ. vs 1 ex. player, PSPACE-complete
- DQBF: three-player game, 1 univ vs 2 ex. players, NEXPTIME-complete [Peterson/Reif '79, Babai et al '91]

Dependencies can make sure that the existential players do not communicate → allows reusing space.

Example: unique existentials indexed by i

$$\forall i, i' \; \exists y(i), y'(i') \; (i = i') \rightarrow (y = y') \wedge (i \neq i') \rightarrow (y \neq y')$$

Minimal Falsity and Quantification

A (D)QCNF formula is minimal false (MF) iff it is false and removing an arbitrary clause makes it true.

MF subformulas of the bound parts determine the role of the free parts in (D)QCNF formulas:

$$Q \bigwedge_{1 \leq i \leq q} (\phi_i^b \vee \phi_i^f) \approx \underbrace{\bigwedge_{(Q\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b) \in S}}_{\text{all MF subformulas of the bound parts}} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f)$$

Minimal Falsity and Quantification

$$Q \bigwedge_{1 \le i \le q} (\phi_i^b \vee \phi_i^f) \approx \bigwedge_{\underbrace{(Q\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b) \in S}_{\text{all MF subformulas of the bound parts}}} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f)$$

Special Case: Minimal false (D)Q2-CNF formulas contain at most two ∃-unit clauses.

Consequence: decompose (D)Q2-CNF^b formulas into subformulas with at most two universals:

$$\Phi \approx \bigwedge_{u_1, u_2 \in \forall var(\Phi)} \Phi|_{u_1, u_2}$$

Universal Expansion in DQ2-CNF^b

(D)Q2-CNF^b $\Phi \approx \bigwedge_{u_1,u_2 \in \forall var(\Phi)} \underbrace{\Phi|_{u_1,u_2}}$

u_1, u_2 can be eliminated by universal expansion with linear formula growth.

→ $\Phi \in$ (D)Q2-CNF^b can be transformed in time $O(|\forall|^2|\Phi|)$ into an equivalent ∃2-CNF^b formula.

→ (D)Q2-CNF^b satisfiability is NP-complete.

→ Special Case: DQ2-CNF* (whole matrix in 2-CNF) is tractable (even in linear time by a modification of the Aspvall/Plass/Tarjan algorithm).

Transforming $\exists$2-CNF^b to $\exists$2-HORN^b

Graph Encoding of ∃2-HORN^b 1/4

The Aspvall/Plass/Tarjan algorithm maps a 2-CNF formula over variables $v_1,\ldots,v_n$ into a graph with $2n$ nodes $v_1,\neg v_1, \ldots, v_n,\neg v_n$. For each clause $(l \vee k)$, it has edges $\neg l \rightarrow k$ and $\neg k \rightarrow l$.

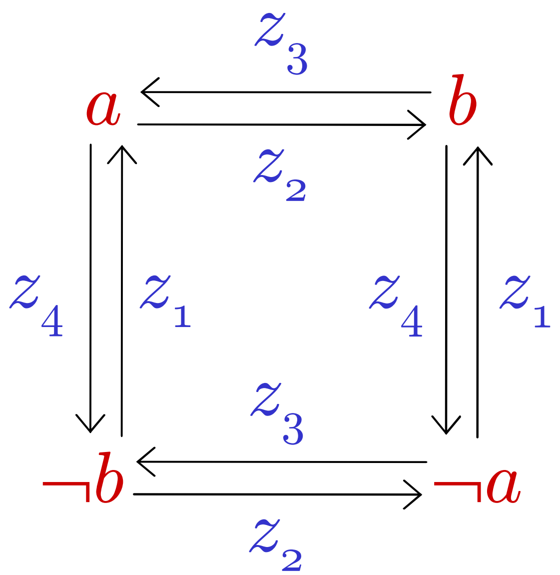
For ∃2-CNF^b, we build this graph for the bound parts and label the edges with the corresponding free parts:

$(l \vee k \vee \phi^f_i)$ gives edges $\neg l \overset{\phi^f_i}{\rightarrow} k$ and $\neg k \overset{\phi^f_i}{\rightarrow} l$.

Graph Encoding of ∃2-HORN^b 2/4

Example:

$\exists a \exists b\ (a \lor b \lor z_1) \land (\neg a \lor b \lor z_2) \land (a \lor \neg b \lor z_3) \land (\neg a \lor \neg b \lor z_4)$

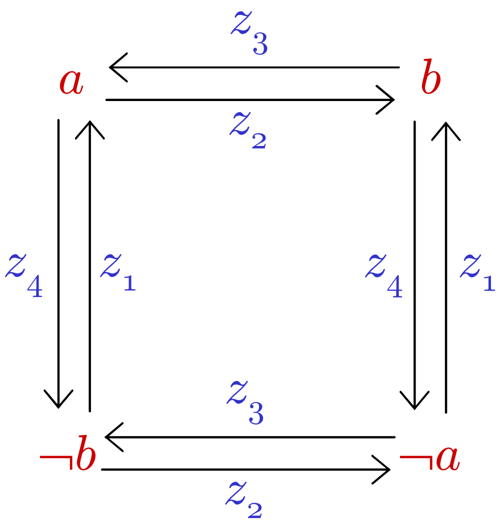


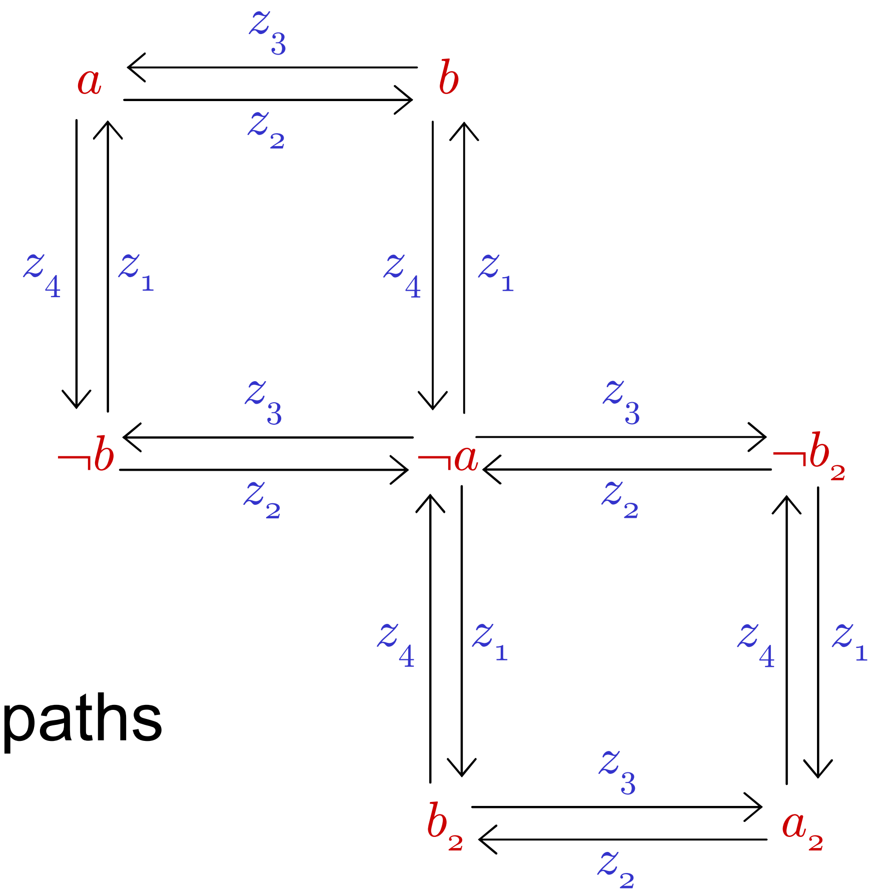
Formula is satisfiable iff there is a truth assignment to the free variables such that all paths from a node v to $\neg v$ and back to v have at least one satisfied edge label.

How to encode this criterion as a ∃2-HORN^b formula?

Graph Encoding of ∃2-HORN^b 3/4

- Consider only one variable v at a time.
- Unfold the graph by mirroring it around $\neg v$.





→ Sufficient to consider simple paths from the source to the sink

Graph Encoding of ∃2-HORN^b 4/4

Checking whether in a graph with nodes $v_1,...,v_n$ all paths from v_1 to v_n have a satisfied edge label is easy to encode in ∃2-HORN^b:

$$\exists v_1 ... \exists v_n \; v_1 \wedge \neg v_n \bigwedge_{(v_i \xrightarrow{\sigma} v_j) \in E} (\neg v_i \vee v_j \vee \sigma)$$

→ Transformation of a ∃2-CNF^b formula Ψ into ∃2-HORN^b in time and space O(|∃|·|Ψ|).

Conclusion

Conclusion

Quantification with universal and existential or even dependency quantifiers can be very powerful, but not in combination with 2-CNF or Horn restrictions, even if only applied to bound variables:

$\text{DQ2-CNF}^b =_{\text{poly-time}} \exists\text{2-CNF}^b =_{\text{poly-time}} \exists\text{2-HORN}^b$

Nevertheless, $\exists\text{2-HORN}^b >_{\text{poly-time}} \text{CNF}$.

Is $\exists\text{HORN}^b$ even more expressive?