Models and Quantifier Elimination for Quantified Horn Formulas

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Abstract

In this paper, quantified Horn formulas ($QHORN$) are investigated. We prove that the behavior of the existential quantifiers depends only on the cases where at most one of the universally quantified variables is zero. Accordingly, we give a detailed characterization of $QHORN$ satisfiability models which describe the set of satisfying truth assignments to the existential variables. We also consider quantified Horn formulas with free variables ($QHORN^*$) and show that they have monotone equivalence models.

The main application of these findings is that any quantified Horn formula Φ of length $|\Phi|$ with free variables, $|\forall|$ universal quantifiers and an arbitrary number of existential quantifiers can be transformed into an equivalent quantified Horn formula of length $O(|\forall| \cdot |\Phi|)$ which contains only existential quantifiers.

We also obtain a new algorithm for solving the satisfiability problem for quantified Horn formulas with or without free variables in time $O(|\forall| \cdot |\Phi|)$ by transforming the input formula into a satisfiability-equivalent propositional formula. Moreover, we show that $QHORN$ satisfiability models can be found with the same complexity.

Key words: quantified Boolean formula, quantified Horn formula, model, quantifier elimination, satisfiability

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1 Introduction

Quantified Boolean formulas (QBF) offer a concise way to represent formulas which arise in areas such as planning, scheduling or verification. The ability to provide compact representations for many Boolean functions does however come at a price: determining the satisfiability of formulas in QBF is PSPACE-complete, which is assumed to be significantly harder than the NP-completeness of the propositional SAT problem. However, continued research and the lifting of propositional SAT techniques to $QBFs$ (see, e.g., [1,2,3]) have recently produced interesting improvements and have led to the emergence of more powerful QBF-SAT solvers [4].

Furthermore, the satisfiability problem is known to be tractable for some restricted subclasses like $QHORN$ [5] or $Q2{-}CNF$ [6]. Those classes are defined by imposing restrictions on the syntactic structure of the formula. In this paper, we will focus on the class of quantified Horn formulas ($QHORN$), which contains all QBF formulas in conjunctive normal form (CNF) whose clauses have at most one positive literal. That means the clauses can be thought of as implications where the premise is a conjunction of positive literals and the conclusion is (at most) one positive literal. Being able to represent this simple version of the "`if-then`" statement in a tractable subclass of QBF is part of the importance of the class $QHORN$. Another important point is that $QHORN$ formulas may occur as subproblems when solving arbitrary QBF formulas [7].

The interesting question which we want to investigate is how such a syntactic restriction affects the structure of the set of satisfying truth assignments to the existentially quantified variables. Knowing about that relationship might allow us to transform a formula into a simplified equivalent formula by dropping or substituting certain quantified variables.

A suitable concept for describing the satisfying truth assignments to the existential variables is the notion of *models* for formulas in QBF, which has been introduced in [8]. A model maps each existential variable y_i to a propositional formula f_{y_i} over universal variables whose quantifiers precede the quantifier of y_i. A model is called a *satisfiability model* if substituting the model functions for the existential variables leads to a formula which is true. Consider a two-person game represented by the QBF formula $\Phi = \forall x_1 \exists y_1 ... \forall x_n \exists y_n \, G(x_1, y_1, ..., x_n, y_n)$, where x_i is the i-th move of the first player and y_j is the j-th move of the second player. The moves are binary, and the function G determines for a given sequence $x_1, y_1, ..., x_n, y_n$ of moves which player wins. Assume $G = 1$ whenever player 2 wins. Then a model describes which moves y_i the second player makes depending on the preceding moves $x_1, ..., x_i$ of player 1. And a satisfiability model describes a winning strategy for player 2, which means that for any sequence of opponent moves $x_1, ..., x_i$, he can find suitable moves y_i such that finally $G(x_1, y_1, ..., x_n, y_n) = 1$.

In this paper, we demonstrate that the special syntactic structure of quantified Horn formulas has a heavy impact on the interplay of universal and existential quantifiers. We can show that not all possible values of the preceding universal variables are relevant for the choice of the existentials. Instead, only certain combinations of values for the universals, which we can describe by a suitable relation $R_\forall$, are sufficient for determining the satisfiability model. In order to characterize the relevant core of the satisfiability model, we introduce the concept of $R_\forall$-partial satisfiability models. We then prove that for $QHORN$ formulas, the partial model can always be extended to a total satisfiability model, so the partial model alone carries all the necessary information about the behavior of the existential variables.

The paper also investigates Horn formulas in which not all variables are bound by quantifiers. When such free variables are allowed, we indicate this with a star $*$ and write $QHORN^*$. Formulas with free variables are different in that their satisfiability is dependent on the values of the free variables, whereas closed formulas are either true or false. Accordingly, we extend the concept of models for closed formulas to formulas with free variables and investigate which of the structural properties of satisfiability models for closed $QHORN$ formulas are preserved. We prove that those generalized models are monotone.

The special behavior of the quantifiers has far-reaching consequences. We present the following results:

- All the universal quantifiers in a $QHORN^*$ formula can be eliminated in quadratic time and with only quadratic blowup of the formula. To be more precise, we present an algorithm which transforms any formula $\Phi \in QHORN^*$ of length $|\Phi|$ with free variables, $|\forall|$ universal quantifiers and an arbitrary number of existential quantifiers into an equivalent quantified Horn formula of length $O(|\forall| \cdot |\Phi|)$ which contains only existential quantifiers.
- We obtain a new algorithm for solving $QHORN^*$-SAT in time $O(|\forall| \cdot |\Phi|)$ by transforming the input formula into a satisfiability-equivalent propositional formula.
- We show how to find satisfiability models for $QHORN$ formulas in time $O(|\forall| \cdot |\Phi|)$, which means finding models is just as difficult as determining satisfiability.

2 Preliminaries

In this section, we recall the basic concepts and terminology for propositional formulas and QBF. We also introduce some additional notation.

A literal is a propositional variable (v) or a negated variable ($\neg v$). A disjunction of literals is called a clause, and a conjunction of clauses is a *CNF* formula.

Quantified Boolean formulas introduce quantifiers over variables. $\forall x\ \phi(x)$ is defined to be true if and only if $\phi(0)$ is true *and* $\phi(1)$ is true. Variables which are bound by universal quantifiers are called universal variables and are usually given the names $x_1, ..., x_n$. Similarly, $\exists y\ \phi(y)$ is defined to be true if and only if $\phi(0)$ *or* $\phi(1)$ is true. In this case, y is called an existential variable. Those usually have names $y_1, ..., y_m$. A quantified Boolean formula Φ is in *prenex form* if $\Phi = Q_1 v_1 ... Q_n v_n\, \phi(v_1, ..., v_n)$ with quantifiers $Q_i \in \{\forall, \exists\}$ and a propositional formula $\phi(v_1, ..., v_n)$ over variables $v_1, ..., v_n$. We call ϕ the *matrix* of Φ. Unless mentioned otherwise, we assume that QBF formulas are always in prenex form.

Variables which are not bound by quantifiers are *free* variables. Formulas without free variables are called *closed.* If free variables are allowed, we indicate this with an additional star * after the name of the formula class. Accordingly, QBF is the class of closed quantified Boolean formulas, and QBF^* denotes the quantified Boolean formulas with free variables (and analogously for $QHORN$ and $QHORN^*$, etc.). We write $\Phi(z_1, ..., z_r) = Q\ \phi(z_1, ..., z_r)$ or $\Phi(\mathbf{z}) = Q\ \phi(\mathbf{z})$ for a QBF^* formula with prefix Q, matrix ϕ and free variables $\mathbf{z} = (z_1, ..., z_r)$.

A closed QBF formula is either true or false. It is true if there exists an assignment of truth values to the existential variables depending on the preceding universal variables such that the propositional matrix of the formula is true for all values of the universal variables. For example, $\Phi = \forall x \exists y\ (\neg x \vee y) \wedge (x \vee \neg y)$ is true, because when choosing $y = x$, the resulting matrix $(\neg x \vee x) \wedge (x \vee \neg x)$ is tautological.

The truth value of a QBF^* formula depends on the value of the free variables. A QBF^* formula is satisfiable for a given truth assignment $t(\mathbf{z}) := (t(z_1), ..., t(z_r)) \in \{0, 1\}^r$ to the free variables $\mathbf{z} = (z_1, ..., z_r)$ if there exists an assignment of truth values to the existential variables depending on the free variables and the preceding universal variables such that the matrix of the formula is true for all values of the universal variables. For example, $\Phi(z) = \forall x \exists y\ (x \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg y \vee z)$ is satisfiable for $z = 1$, because when choosing $y = \neg x$, the resulting matrix $(x \vee \neg x) \wedge (\neg x \vee x) \wedge (x \vee z)$ is tautological for $z = 1$. For $z = 0$, however, Φ is unsatisfiable, because we cannot find a suitable y.

The concept of satisfiability models as a means for describing the satisfying truth assignments to the existential variables is essential for this paper, so we provide a formal definition (based on [8]):

Definition 1 *For a quantified Boolean formula $\Phi \in QBF$ with existential variables $\mathbf{y} = (y_1, ..., y_m)$, let $M = (f_{y_1}, ..., f_{y_m})$ be a mapping which associates with each existential variable y_i a propositional formula f_{y_i} over universal variables whose quantifiers precede the quantifier of y_i. Then M is a* ***satisfiability model*** *for Φ if the resulting formula $\Phi[\mathbf{y}/M] := \Phi[y_1/f_{y_1}, ..., y_m/f_{y_m}]$, where simultaneously each existential variable y_i is replaced by its corresponding formula f_{y_i} and the existential quantifiers are dropped from the prefix, is true.*

In Section 6, we will investigate how this concept can be extended to formulas with free variables.

Two QBF^* formulas $\Psi_1(z_1, ..., z_r)$ and $\Psi_2(z_1, ..., z_r)$ are said to be *equivalent* ($\Psi_1 \approx \Psi_2$) if and only if $\Psi_1 \models \Psi_2$ and $\Psi_2 \models \Psi_1$, where *semantic entailment* $\models$ is defined as follows: $\Psi_1 \models \Psi_2$ if and only if for all truth assignments $t(\mathbf{z}) = (t(z_1), ..., t(z_r)) \in \{0, 1\}^r$ to the free variables $\mathbf{z} = (z_1, ..., z_r)$, we have $\Psi_1(t(\mathbf{z})) = 1 \;\Rightarrow\; \Psi_2(t(\mathbf{z})) = 1$.

We need some additional notation:

For $\Phi \in QBF^*$, $\Phi(\mathbf{z}) = Q\,\phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$, we introduce the following notation to combine successive quantifiers of the same kind:
If Q has the form $Q = \forall x_{1,1}...\forall x_{1,n_1}\exists y_{1,1}...\exists y_{1,m_1} ... \forall x_{r,1}...\forall x_{r,n_r}\exists y_{r,1}...\exists y_{r,m_r}$ with $n_i \geq 1$ and $m_i \geq 1$ for $i = 1, .., r$, we simply write $Q = \forall X_1 \exists Y_1 ... \forall X_r \exists Y_r$ with quantifier blocks $X_i = (x_{i,1}, ..., x_{i,n_i})$ and $Y_i = (y_{i,1}, ..., y_{i,m_i})$, $i = 1, .., r$.

Another notation that we use is $AB := (a_1, ..., a_m, b_1, ..., b_n)$ to denote the concatenation of two tuples $A = (a_1, ..., a_m)$ and $B = (b_1, ..., b_n)$.

3 Eliminating Universal Quantifiers

It is known that converting a quantified Horn formula into an equivalent propositional Horn formula may result in an exponentially longer formula (see, e.g., [9]), so this is not a practical way to go. The question then is whether it is at least possible to eliminate just one kind of quantifier. Can we remove all the universal quantifiers and leave only existential ones, such that the length of the resulting existentially quantified Horn formula is bounded by a polynomial? In this section, we investigate the role of the universal quantifiers and then use that knowledge to eliminate them. Let us begin with the following definition:

Definition 2 *A formula $\Phi \in QHORN^*$ is an* ***existentially quantified Horn formula*** *with free variables if it is of the form $\Phi(\mathbf{z}) = \exists y_1 ... \exists y_m\, \phi(\mathbf{z})$ ($m \geq 0$), i.e. if it does not contain universally quantified variables. The class of all such formulas we denote by $\exists HORN^*$.*

The goal of the following investigation is to transform an arbitrary formula in $QHORN^*$ into an equivalent formula in $\exists HORN^*$ with a polynomial increase in length. The method that we present is a specialization of the known exponential method of expanding universal quantifiers in general QBF^* formulas in *CNF*. We first present the general technique and then investigate the methodology for refining it in the special case of $QHORN^*$ formulas.

3.1 Eliminating Universal Quantifiers in QBF^* Formulas

In QBF^*, quantifiers can be considered as abbreviations. We have the equivalence $\exists y\ \Phi(y, \mathbf{z}) \approx \Phi(0, \mathbf{z}) \vee \Phi(1, \mathbf{z})$ (the QBF^* analog of the well-known Shannon Expansion) and the dual $\forall x\ \Phi(x, \mathbf{z}) \approx \Phi(0, \mathbf{z}) \wedge \Phi(1, \mathbf{z})$. This can be used to eliminate quantifiers by expansion. Since we have *CNF* formulas, universal expansion is more convenient as it retains the *CNF* structure. The general method for expanding a universal quantifier is rather straightforward: two copies of the original matrix are generated, one for the universally quantified variable being true, and one for that variable being false. Since $(\exists y\ \Phi(0, y)) \wedge (\exists y\ \Phi(1, y)) \not\approx \exists y\ (\Phi(0, y) \wedge \Phi(1, y))$, existential variables which are in the scope of that universal quantifier need to be duplicated as well. For example, in the formula $\exists y_1 \forall x \exists y_2\, \phi(x, y_1, y_2)$, the choice for the existential variable y_2 depends on the value of x. We must therefore introduce two separate instances $y_2^{(0)}$ and $y_2^{(1)}$ of the original variable y_2, where $y_2^{(0)}$ is used in the copy of the matrix for $x = 0$, and analogously $y_2^{(1)}$ for $x = 1$. We obtain the expanded formula $\exists y_1 \exists y_2^{(0)} \exists y_2^{(1)}\, \phi(0, y_1, y_2^{(0)}) \wedge \phi(1, y_1, y_2^{(1)})$. For multiple universal quantifiers, we successively expand each universal quantifier, starting with the innermost.

Based on this informal description, we now provide a formal representation of the expanded formula.
Let $\Phi \in QBF^*$ with $\Phi(\mathbf{z}) = \forall X_1 \exists Y_1 ... \forall X_r \exists Y_r\, \phi(X_1, ..., X_r, Y_1, ..., Y_r, \mathbf{z})$ be the formula whose universal quantifiers we want to expand. $X_i = (x_{i,1}, ..., x_{i,n_i})$ and $Y_i = (y_{i,1}, ..., y_{i,m_i})$ ($n_i \geq 1$ and $m_i \geq 1$, $i = 1, ..., r$, $r \geq 1$) are the quantifier blocks in the prefix, and ϕ is the propositional matrix in *CNF*. Without loss of generality, we assume that the outermost quantifiers are universal. If they were existential, we could treat these existentially quantified variables as free variables, and the outermost quantifiers in the remaining prefix would then be universal. Furthermore, we assume that the innermost quantifiers are existential, as universal variables which do not dominate any existential vari-

ables can be removed.

The expanded formula is then given as

$$\begin{aligned}\Phi_{\exists\mathrm{exp}}(\mathbf{z}) := &\bigwedge_{A_1\in\{0,1\}^{n_1}} \Big(\exists Y_1^{A_1}\\ &\bigwedge_{A_2\in\{0,1\}^{n_2}} \Big(\exists Y_2^{A_1A_2}\\ &\quad\vdots\\ &\bigwedge_{A_r\in\{0,1\}^{n_r}} \Big(\exists Y_r^{A_1...A_r}\, \phi(A_1...A_r, Y_1^{A_1}...Y_r^{A_1...A_r}, \mathbf{z})\Big)\, ...\Big)\Big)\end{aligned}$$

The tuples A_i represent the possible truth assignments to the universal variables $x_{i,1}, ..., x_{i,n_i}$. The expression $\bigwedge_{A_i\in\{0,1\}^{n_i}}$ should be understood as a conjunction of 2^{n_i} clauses, one for each truth assignment. Finally, $\exists Y_i^{A_1..A_i}$ is an abbreviation for $\exists y_{i,1}^{A_1...A_i}...\exists y_{i,m_i}^{A_1...A_i}$, the copies of the i-th block of existential quantifiers. The additional index $A_1...A_i$ is used to tag each copy with the values of the preceding universal variables. Its purpose is to have a unique name for each of those copies. For example, four copies of $y_{i,j}$ would be named $y_{i,j}^{(0,0)}$, $y_{i,j}^{(0,1)}$, $y_{i,j}^{(1,0)}$ and $y_{i,j}^{(1,1)}$.

Using induction on the number of blocks of universal quantifiers, it is possible to show that $\Phi(\mathbf{z}) \approx \Phi_{\exists\mathrm{exp}}(\mathbf{z})$. We omit this proof, as it is quite obvious that $\Phi_{\exists\mathrm{exp}}$ is simply the formalization of the elimination algorithm described above.

Here is an example: the formula

$$\Phi(\mathbf{z}) = \forall x_1 \exists y_1 \forall x_2 \forall x_3 \exists y_2\, \phi(x_1, x_2, x_3, y_1, y_2, \mathbf{z})$$

is expanded to $\Phi_{\exists\mathrm{exp}}(\mathbf{z}) =$

$$\begin{aligned}&\exists y_1^{(0)}(\exists y_2^{(0,0,0)}\phi(0,0,0,y_1^{(0)},y_2^{(0,0,0)},\mathbf{z}) \wedge \exists y_2^{(0,0,1)}\phi(0,0,1,y_1^{(0)},y_2^{(0,0,1)},\mathbf{z}) \wedge\\ &\qquad \exists y_2^{(0,1,0)}\phi(0,1,0,y_1^{(0)},y_2^{(0,1,0)},\mathbf{z}) \wedge \exists y_2^{(0,1,1)}\phi(0,1,1,y_1^{(0)},y_2^{(0,1,1)},\mathbf{z}))\wedge\\ &\exists y_1^{(1)}(\exists y_2^{(1,0,0)}\phi(1,0,0,y_1^{(1)},y_2^{(1,0,0)},\mathbf{z}) \wedge \exists y_2^{(1,0,1)}\phi(1,0,1,y_1^{(1)},y_2^{(1,0,1)},\mathbf{z}) \wedge\\ &\qquad \exists y_2^{(1,1,0)}\phi(1,1,0,y_1^{(1)},y_2^{(1,1,0)},\mathbf{z}) \wedge \exists y_2^{(1,1,1)}\phi(1,1,1,y_1^{(1)},y_2^{(1,1,1)},\mathbf{z}))\end{aligned}$$

$\Phi_{\exists\mathrm{exp}}$ is not in prenex form. This would be easy to fix by moving all quantifiers to the front. In the sample formula above, the prefix might then look like

$$\exists y_1^{(0)}\exists y_2^{(0,0,0)}\exists y_2^{(0,0,1)}\exists y_2^{(0,1,0)}\exists y_2^{(0,1,1)}\exists y_1^{(1)}\exists y_2^{(1,0,0)}\exists y_2^{(1,0,1)}\exists y_2^{(1,1,0)}\exists y_2^{(1,1,1)}\,.$$

For clarity's sake, we did not consider this in the general formula $\Phi_{\exists\text{exp}}$.

As the expansion example above demonstrates, the resulting formula is rather voluminous. If there are n universal quantifiers in an input formula Φ, its expansion $\Phi_{\exists\text{exp}}$ contains 2^n copies of the formula's original matrix. Therefore, the expansion generally results in an exponential increase in length.

In combination with other techniques like Q-Resolution [10] or by expanding only a limited number of universals [11], the rapid growth of the formula can often be mitigated, making the method quite successful in practice. Nevertheless, it remains problematic for larger input formulas. But we can significantly simplify the expansion in the special case of quantified Horn formulas.

3.2 Partial Satisfiability Models

In this section, we show that for quantified Horn formulas, we do not need to consider all possible truth assignments to the universal variables. We restrict those assignments according to a relation $R_\forall(n)$ on the set of possible truth assignments to n universals.

Definition 3 *By B_n^i, we denote the bit vector of length n where only the i-th element is zero, i.e. $B_n^i := (b_1, ..., b_n)$ with $b_i = 0$ and $b_j = 1$ for $j \neq i$.*

Moreover, we define the following relations on n-tuples of truth values:

(1) $Z_{\leq 1}(n) = \bigcup_i \{B_n^i\} \,\cup\, \{(1, ..., 1)\}$ *(at most one zero)*
(2) $Z_{=1}(n) = \bigcup_i \{B_n^i\}$ *(exactly one zero)*
(3) $Z_{\geq 1}(n) = \{(a_1, ..., a_n) \mid \exists i : a_i = 0\}$ *(at least one zero)*

For example, if $n = 3$, we have the following relations:

$$
\begin{aligned}
Z_{\leq 1}(3) &= \{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\} \\
Z_{=1}(3) &= \{(0,1,1),(1,0,1),(1,1,0)\} \\
Z_{\geq 1}(3) &= \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)\}
\end{aligned}
$$

We omit the parameter n and simply write $Z_{\leq 1}$ (or $Z_{=1}$ resp. $Z_{\geq 1}$) when it is clear from the context. Usually, n equals the total number of the universal quantifiers in a given formula.

Let $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in QBF$. The definition of a satisfiability model in Section 2 requires that substituting the existentials $\mathbf{y}$ in Φ produces a formula $\Phi[\mathbf{y}/M]$ which is true. That means the matrix $\phi[\mathbf{y}/M]$ must be true for all possible assignments to the universals $\mathbf{x}$. We now introduce a special kind of satisfiability model which weakens this condition: a so-called *$R_\forall$-partial satisfiability model* is only required to satisfy $\phi[\mathbf{y}/M]$ for certain truth assignments to the universal variables which are given by a relation $R_\forall$.

Definition 4 *For $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in QBF$ with universals $\mathbf{x} = (x_1, ..., x_n)$ and existentials $\mathbf{y} = (y_1, ..., y_m)$, let $M = (f_{y_1}, ..., f_{y_m})$ be a mapping which associates with each existential variable y_i a propositional formula f_{y_i} over universal variables whose quantifiers precede the quantifier of y_i. Furthermore, let $R_\forall(n)$ be a relation on the set of possible truth assignments to the n universals. Then M is a $R_\forall$-**partial satisfiability model** for Φ if the formula $\phi[\mathbf{y}/M]$ is true for all $\mathbf{x} \in R_\forall(n)$.*

For the sake of completeness, we also allow $n = 0$ (i.e. formulas without universal variables) in the above definition, in which case the f_{y_i} are constants 0 or 1, and we require that $\phi[\mathbf{y}/M]$ is true.

It is important to point out that satisfiability models (and thus also partial satisfiability models and the related results presented in this section) are only defined for closed formulas, i.e. for formulas without free variables. Nevertheless, this concept is also important for the general case with free variables, because we often consider fixed assignments to the free variables and can then proceed as in the closed case. Section 3.3 will give a nice demonstration of this approach.

Consider the following example: the formula $\Phi = \forall x_1 \forall x_2 \exists y\, (x_1 \vee y) \wedge (x_2 \vee \bar{y})$ does not have a satisfiability model, but $M = (f_y)$ with $f_y(x_1, x_2) = \bar{x}_1 \vee x_2$ is a $Z_{\leq 1}$-partial satisfiability model for Φ, because $\phi[y/M] = (x_1 \vee \bar{x}_1 \vee x_2) \wedge (x_2 \vee (x_1 \wedge \bar{x}_2)) \approx x_2 \vee x_1$, which is true for all $\mathbf{x} = (x_1, x_2)$ with $\mathbf{x} \in Z_{\leq 1}$.

It is not surprising that the mere existence of a $Z_{\leq 1}$-partial satisfiability model does not imply the existence of a (total) satisfiability model - at least not in the general case. Interestingly, this implication is indeed true for quantified Horn formulas. We now show: if we can find a $Z_{\leq 1}$-partial satisfiability model M to satisfy a quantified Horn formula whenever at most one of the universals is false, then we can also satisfy the formula for arbitrary truth assignments to the universals. We achieve this by using M to construct a (total) satisfiability model M^t.

Definition 5 *Let $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in QHORN$ be a quantified Horn formula with universal variables $\mathbf{x} = (x_1, ..., x_n)$ and existentials $\mathbf{y} = (y_1, ..., y_m)$, and let $M = (f_{y_1}, .., f_{y_m})$ be a $Z_{\leq 1}$-partial satisfiability model for Φ. For each*

$f_{y_i}(x_1, ..., x_{n_i})$ in M, we define $f^t_{y_i}$ as follows:

$$\begin{aligned} f^t_{y_i}(x_1, ..., x_{n_i}) := \quad & (x_1 \ \ \vee f_{y_i}(0, 1, 1, ..., 1)) \\ \wedge\ & (x_2 \ \ \vee f_{y_i}(1, 0, 1, ..., 1)) \\ \wedge\ & ... \\ \wedge\ & (x_{n_i} \vee f_{y_i}(1, 1, ..., 1, 0)) \\ \wedge\ & f_{y_i}(1, ..., 1) \end{aligned}$$

*Then we call $M^t = (f^t_{y_1}, ..., f^t_{y_m})$ the **total completion** of M.*

Please notice that the previous definition is equivalent to the following:

$$\begin{aligned} f^t_{y_i}(x_1, ..., x_{n_i}) = \quad & (\bar{x}_1 \ \ \rightarrow f_{y_i}(0, 1, 1, ..., 1)) \\ \wedge\ & (\bar{x}_2 \ \ \rightarrow f_{y_i}(1, 0, 1, ..., 1)) \\ \wedge\ & ... \\ \wedge\ & (\bar{x}_{n_i} \rightarrow f_{y_i}(1, 1, ..., 1, 0)) \\ \wedge\ & f_{y_i}(1, ..., 1) \end{aligned}$$

When some of the arguments are zero, consider *all cases* where *at most one* of those arguments is zero and return the conjunction of the corresponding original function values. For example, $f^t_y(1, 0, 0, 1) = f_y(1, 0, 1, 1) \wedge f_y(1, 1, 0, 1) \wedge f_y(1, 1, 1, 1)$. In case all the arguments are 1, simply return the value of the original function, i.e. $f^t_y(1, .., 1) = f_y(1, .., 1)$. These observations lead to the following lemma:

Lemma 6 *Let $t(\mathbf{x}) = (t(x_1), ..., t(x_n)) \in Z_{\geq 1}(n)$ with $t(x_{z_1}) = 0, ..., t(x_{z_k}) = 0$ and $t(x_s) = 1$ for $s \neq z_1, ..., z_k$ be a truth assignment to the universal variables where $k \geq 1$ universals $x_{z_1}, ..., x_{z_k}$ are zero. Then the definition of $f^t_{y_i}$ implies*

$$f^t_{y_i}(t(x_1), ..., t(x_{n_i})) = \bigwedge_{1 \leq j \leq k} f_{y_i}(t_{z_j}(x_1), ..., t_{z_j}(x_{n_i})) \wedge f_{y_i}(1, ..., 1)$$

where $t_{z_j}(\mathbf{x}) = (t_{z_j}(x_1), ..., t_{z_j}(x_n)) = B^{z_j}_n$ is a truth assignment where exactly one universal x_{z_j} is zero.

Moreover, total expansion equals the partial model when all universals on which y_i depends are 1:

$$f^t_{y_i}(1, ..., 1) = f_{y_i}(1, ..., 1)$$

This definition is based on an observation: it is a well known fact about *propositional* Horn formulas, proved by Alfred Horn himself [12], that the intersection of two satisfying truth assignments is a satisfying truth assignment, too. Let $t_1(\mathbf{x}) = (t_1(x_1), ..., t_1(x_n)) \in \{0,1\}^n$ and $t_2(\mathbf{x}) = (t_2(x_1), ..., t_2(x_n)) \in \{0,1\}^n$ be two truth assignments over variables $x_1, ..., x_n$, then the intersection of t_1 and t_2 is defined as

$$t_1(\mathbf{x}) \cap t_2(\mathbf{x}) = (t_1(x_1) \wedge t_2(x_1), ..., t_1(x_n) \wedge t_2(x_n)) \ .$$

Our idea is to establish a similar relationship between the satisfying truth assignments to the existential variables in a quantified Horn formula, taking also into consideration the universally quantified variables. Assume that a *QHORN* formula with two universal variables x_i and x_j is known to be satisfiable when $x_i = 0$ and $x_j = 1$ or when $x_i = 1$ and $x_j = 0$. That means there exist two truth assignments t_1 and t_2 to the existential variables such that the formula is satisfied in both cases. If we lift the closure under intersection to the quantified case, it means that the intersection of t_1 and t_2 satisfies the formula when both x_i and x_j are zero.

An important point to consider is that we have to obey the quantifier dependencies when choosing truth values for the existential variables. Assume the previous example includes an existential variable y_k with $t_1(y_k) = 1$ and $t_2(y_k) = 0$ and the additional restriction that $\exists y_k$ occurs earlier in the prefix than $\forall x_j$. Then y_k does not depend on x_j, but the intersection of t_1 and t_2 would assign y_k the value 0 when $x_i = 0$ and $x_j = 0$, which is not allowed, because we have already set y_k to 1 when $x_i = 0$ (but $x_j = 1$). This shows that intersecting arbitrary satisfying truth assignments is not appropriate for *QHORN* formulas. However, the proof of Theorem 7 guarantees by construction that quantifier dependencies are respected. Another point to notice is that we always intersect with $f_{y_i}(1, ..., 1)$. This makes sure that we reduce $f^t_{y_i}$ to a well-defined value from the partial satisfiability model in cases where all zeros are assigned to universals on which y_i does not depend.

Theorem 7 *Let $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in QHORN$ be a quantified Horn formula with a $Z_{\leq 1}$-partial satisfiability model $M = (f_{y_1}, .., f_{y_m})$. Then the total completion of M, i.e. $M^t = (f^t_{y_1}, ..., f^t_{y_m})$ as defined above, is a satisfiability model for Φ.*

Proof: We must show that $\phi[\mathbf{y}/M^t]$ is true for all truth assignments to the universal variables. Since $f^t_{y_i}(1, ..., 1) = f_{y_i}(1, ..., 1)$, we only need to consider truth assignments where at least one universal is zero.

Let $t(\mathbf{x}) = (t(x_1), ..., t(x_n)) \in Z_{\geq 1}(n)$ with $t(x_{z_1}) = 0$, ..., $t(x_{z_k}) = 0$ and $t(x_s) = 1$ for $s \neq z_1, ..., z_k$ be a truth assignment to the universal variables where $k \geq 1$ universals x_{z_1}, ..., x_{z_k} are zero. When we combine the truth assignment to the universals and the corresponding values of the model functions into a $(n+m)$-tuple of truth values, we obtain the following bit vector:

$$\tau = (t(x_1), ..., t(x_n), f^t_{y_1}(t(x_1), ..., t(x_{n_1})), ..., f^t_{y_m}(t(x_1), ..., t(x_{n_m})))$$

Our goal is to prove that the propositional matrix ϕ is true under the truth value assignment $\tau = (\tau(x_1), ..., \tau(x_n), \tau(y_1), ..., \tau(y_m))$. We can write the tuple $t(\mathbf{x})$ with k universals being zero as an intersection $t(\mathbf{x}) = t_{z_1}(\mathbf{x}) \cap ... \cap t_{z_k}(\mathbf{x})$ of k assignments $t_{z_j}(\mathbf{x}) = B^{z_j}_n$ with exactly one zero each. Similar to the definition of $f^t_{y_i}$, it is useful to intersect with $(1, .., 1)$ as well. With this trick, we have $t(\mathbf{x}) = t_{z_1}(\mathbf{x}) \cap ... \cap t_{z_k}(\mathbf{x}) \cap (1, ..., 1)$ and can decompose τ as follows:

$$\begin{aligned}
\tau = \;& (t_{z_1}(\mathbf{x}), f^t_{y_1}(t(\mathbf{x}_{1..n_1})), ..., f^t_{y_m}(t(\mathbf{x}_{1..n_m}))) \\
\cap \;& \cdots \\
\cap \;& (t_{z_k}(\mathbf{x}), f^t_{y_1}(t(\mathbf{x}_{1..n_1}), ..., f^t_{y_m}(t(\mathbf{x}_{1..n_m}))) \\
\cap \;& (1, ..., 1, f^t_{y_1}(t(\mathbf{x}_{1..n_1}), ..., f^t_{y_m}(t(\mathbf{x}_{1..n_m})))
\end{aligned}$$

For clarity, we abbreviate $t(\mathbf{x}_{1..n_i}) := (t(x_1), ..., t(x_{n_i}))$ and $f(\mathbf{1}) := f(1, ..., 1)$. Now, Lemma 6 allows us to decompose this even further:

$$\begin{aligned}
\tau = \;& (t_{z_1}(\mathbf{x}), \bigwedge_{j=1..k} f_{y_1}(t_{z_j}(\mathbf{x}_{1..n_1})) \wedge f_{y_1}(\mathbf{1}), ..., \bigwedge_{j=1..k} f_{y_m}(t_{z_j}(\mathbf{x}_{1..n_m})) \wedge f_{y_m}(\mathbf{1})) \\
\cap \;& \cdots \\
\cap \;& (t_{z_k}(\mathbf{x}), \bigwedge_{j=1..k} f_{y_1}(t_{z_j}(\mathbf{x}_{1..n_1})) \wedge f_{y_1}(\mathbf{1}), ..., \bigwedge_{j=1..k} f_{y_m}(t_{z_j}(\mathbf{x}_{1..n_m})) \wedge f_{y_m}(\mathbf{1})) \\
\cap \;& (1, ..., 1, \bigwedge_{j=1..k} f_{y_1}(t_{z_j}(\mathbf{x}_{1..n_1})) \wedge f_{y_1}(\mathbf{1}), ..., \bigwedge_{j=1..k} f_{y_m}(t_{z_j}(\mathbf{x}_{1..n_m})) \wedge f_{y_m}(\mathbf{1}))
\end{aligned}$$

This can be simplified by distributing the conjunctions over the intersections:

$$\begin{aligned}
\tau = \;& (t_{z_1}(\mathbf{x}), f_{y_1}(t_{z_1}(\mathbf{x}_{1..n_1})), ..., f_{y_m}(t_{z_1}(\mathbf{x}_{1..n_m}))) \\
\cap \;& \cdots \\
\cap \;& (t_{z_k}(\mathbf{x}), f_{y_1}(t_{z_k}(\mathbf{x}_{1..n_1}), ..., f_{y_m}(t_{z_k}(\mathbf{x}_{1..n_m}))) \\
\cap \;& (1, ..., 1, f_{y_1}(\mathbf{1}), ..., f_{y_m}(\mathbf{1})))
\end{aligned}$$

We have split $\tau = (\tau(x_1), ..., \tau(x_n), \tau(y_1), ..., \tau(y_m))$ into an intersection $\tau = \tau_1 \cap ... \cap \tau_k \cap \tau_0$ of $k+1$ individual truth assignments to the universal and existential variables in ϕ. A close look reveals that each τ_i represents a situation where at most one universal is zero and each existential y_i is chosen as determined by f_{y_i} for that constellation of the universals. Under the assumption that $M = (f_{y_1}, .., f_{y_m})$ is a $Z_{\leq 1}$-partial satisfiability model of Φ, we know that ϕ is true under each of those assignments $\tau_0, ..., \tau_k$. Since ϕ is a propositional Horn formula, the intersection of satisfying truth assignments is again a satisfying truth assignment.

By construction, quantifier dependencies are respected, i.e. an existential cannot obtain a different value when only a universal on which it does not depend changes. To see this, we write τ as $\tau = (\tau_1 \cap \tau_0) \cap (\tau_2 \cap \tau_0) \cap ... \cap (\tau_k \cap \tau_0)$. Intersecting τ_i with τ_0 may change the truth value of an existential, but the value of all universals stays the same. And in the outer intersections, the truth value of an existential can only change if one of the universals on which it depends changes value as well. $\Box$

Using Definition 5 and Theorem 7, we can immediately obtain a (total) satisfiability model upon finding a $Z_{\leq 1}$-partial satisfiability model for a quantified Horn formula. This means that the behavior of the existential quantifiers is completely determined by the cases where at most one of the universal variables is false. The cases where more than one of them is assigned false are not relevant for predicting the behavior of the existentials.

On the basis of this interesting result, we now present a transformation which eliminates the universal quantifiers from a quantified Horn formula without significantly increasing its length.

3.3 Eliminating Universal Quantifiers in QHORN Formulas*

Definition 8 *Let $\Phi \in QHORN^*$ with*

$$\Phi(\mathbf{z}) = \forall X_1 \exists Y_1 ... \forall X_r \exists Y_r\, \phi(X_1, ..., X_r, Y_1, ..., Y_r, \mathbf{z})$$

where $X_i = (x_{i,1}, ..., x_{i,n_i})$ and $Y_i = (y_{i,1}, ..., y_{i,m_i})$ ($n_i \geq 1$ and $m_i \geq 1$, $i = 1, ..., r$, $r \geq 1$), be a quantified Horn formula whose outermost quantifiers are universal and whose innermost quantifiers are existential.

Then we define the formula $\Phi_{\exists \mathrm{poly}}(\mathbf{z})$ as

$$\Phi_{\exists \mathrm{poly}}(\mathbf{z}) := \bigwedge_{A_1 \in Assign_1} \Big(\exists Y_1^{A_1} \bigwedge_{A_2 \in Assign_2(A_1)} \Big(\exists Y_2^{A_1 A_2} \cdots \bigwedge_{A_r \in Assign_r(A_1 ... A_{r-1})} \Big(\exists Y_r^{A_1 ... A_r}\, \phi(A_1 ... A_r, Y_1^{A_1} ... Y_r^{A_1 ... A_r}, \mathbf{z}) \Big) ... \Big) \Big)$$

with the restricted set of possible assignments

$$Assign_1 = Z_{\leq 1}(n_1)$$

$$Assign_i(A_1, ..., A_{i-1}) = \begin{cases} Z_{\leq 1}(n_i) & \text{, if } A_1 ... A_{i-1} = \{1\}^{n_1 + ... + n_{i-1}} \\ \{1\}^{n_i} & \text{, else} \end{cases}$$

The only difference between the formula $\Phi_{\exists\text{poly}}$ and the expansion $\Phi_{\exists\text{exp}}$ for general QBF^* formulas which was presented in Section 3.1 is that for quantified Horn formulas, not all possible truth assignments to the universally quantified variables have to be considered. For Horn formulas, we discard assignments where more than one universal variable is false.

For the formula

$$\Phi(\mathbf{z}) = \forall x_1 \exists y_1 \forall x_2 \forall x_3 \exists y_2 \, \phi(x_1, x_2, x_3, y_1, y_2, \mathbf{z})$$

from the example in Section 3.1, we have

$$\begin{aligned} \Phi_{\exists\text{poly}}(\mathbf{z}) = \exists y_1^{(0)} \quad & \exists y_2^{(0,1,1)} \phi(0, 1, 1, y_1^{(0)}, y_2^{(0,1,1)}, \mathbf{z}) \wedge \\ \exists y_1^{(1)} (\; & \exists y_2^{(1,0,1)} \phi(1, 0, 1, y_1^{(1)}, y_2^{(1,0,1)}, \mathbf{z}) \wedge \\ & \exists y_2^{(1,1,0)} \phi(1, 1, 0, y_1^{(1)}, y_2^{(1,1,0)}, \mathbf{z}) \wedge \\ & \exists y_2^{(1,1,1)} \phi(1, 1, 1, y_1^{(1)}, y_2^{(1,1,1)}, \mathbf{z})) \end{aligned}$$

Before we can prove that $\Phi_{\exists\text{poly}}$ is indeed equivalent to Φ, we make a fundamental observation: for the special case that Φ is closed, i.e. there are no free variables, the satisfiability of $\Phi_{\exists\text{poly}}$ implies the existence of a $Z_{\leq 1}$-partial satisfiability model for Φ.

Lemma 9 *Let $\Phi \in QHORN$ be a quantified Horn formula without free variables, and let $\Phi_{\exists\text{poly}}$ be defined as above. If $\Phi_{\exists\text{poly}}$ is satisfiable then Φ has a $Z_{\leq 1}$-partial satisfiability model.*

Proof:

Let t be a satisfying truth assignment to the existentials in $\Phi_{\exists\text{poly}}$. This assignment t provides us with all the information needed to construct a $Z_{\leq 1}$-partial satisfiability model for Φ.

The basic idea is to assemble the truth assignments to the individual copies $y_{i,j}^{(x_{1,1}, ..., x_{i,n_i})}$ of an existential variable $y_{i,j}$ into a common model function. It works as follows: let $y_{i,j}$ be an existential variable in Φ whose corresponding quantifier is preceded by the universal quantifiers $\forall x_{1,1}...\forall x_{i,n_i}$. Then we define:

$$\begin{aligned} f_{y_{i,j}}(x_{1,1},...,x_{i,n_i}) = \quad & (\bar{x}_{1,1} \wedge x_{1,2} \wedge ... \wedge x_{i,n_i} \quad \rightarrow t(y_{i,j}^{(0,1,...,1)})) \\ & \wedge\ (x_{1,1} \wedge \bar{x}_{1,2} \wedge x_{1,3} \wedge ... \wedge x_{i,n_i} \quad \rightarrow t(y_{i,j}^{(1,0,1,...,1)})) \\ & \wedge\ ... \\ & \wedge\ (x_{1,1} \wedge ... \wedge x_{i,n_i-1} \wedge \bar{x}_{i,n_i} \quad \rightarrow t(y_{i,j}^{(1,...,1,0)})) \\ & \wedge\ (x_{1,1} \wedge ... \wedge x_{i,n_i} \quad \rightarrow t(y_{i,j}^{(1,...,1)})) \end{aligned}$$

Now, the $f_{y_{i,j}}$ form a $Z_{\leq 1}$-partial satisfiability model for Φ, because for all $\mathbf{x} = (x_{1,1},...,x_{r,n_r})$ with $\mathbf{x} \in Z_{\leq 1}$, we have $f_{y_{i,j}}(x_{1,1},...,x_{i,n_i}) = t(y_{i,j}^{(x_{1,1},...,x_{i,n_i})})$, and $\phi(x_{1,1},...,x_{r,n_r}, t(y_{1,1}^{(x_{1,1},...,x_{1,n_1})}),...,t(y_{r,m_r}^{(x_{1,1},...,x_{r,n_r})})) = 1$ due to the satisfiability of $\Phi_{\exists\text{poly}}$. □

Using Lemma 9 in combination with Theorem 7, it is now easy to show that $\Phi_{\exists\text{poly}}$ is equivalent to Φ.

Theorem 10 $\Phi_{\exists\text{poly}}$ *is equivalent to* Φ*.*

Proof:

The implication $\Phi(\mathbf{z}) \models \Phi_{\exists\text{poly}}(\mathbf{z})$ is obvious, as the clauses in $\Phi_{\exists\text{poly}}$ are just a subset of the clauses in $\Phi_{\exists\text{exp}}$, which in turn is equivalent to Φ.

The implication $\Phi_{\exists\text{poly}}(\mathbf{z}) \models \Phi(\mathbf{z})$ is more interesting. Assume that $\Phi_{\exists\text{poly}}(\mathbf{z}^*)$ is satisfiable for some fixed $\mathbf{z}^*$. With the free variables fixed, we can treat both $\Phi_{\exists\text{poly}}(\mathbf{z}^*)$ and $\Phi(\mathbf{z}^*)$ as closed formulas and apply Lemma 9 and the results from Section 3.2 as follows:
By Lemma 9, the satisfiability of $\Phi_{\exists\text{poly}}(\mathbf{z}^*)$ implies that $\Phi(\mathbf{z}^*)$ has a $Z_{\leq 1}$-partial satisfiability model. On this partial model, we apply the total expansion from Definition 5 and Theorem 7 to obtain a (total) satisfiability model. The fact that $\Phi(\mathbf{z}^*)$ has a satisfiability model implies that $\Phi(\mathbf{z}^*)$ is satisfiable. □

In the definition of $\Phi_{\exists\text{poly}}$, we can observe that there is one instantiation of the matrix of the original formula for each possible assignment to the universal variables in which either all of those variables are true or exactly one of them is false. There are $n+1$ such assignments. Furthermore, the previous theorem has shown that $\Phi_{\exists\text{poly}}$ is equivalent to Φ, so we have the following corollary.

Corollary 11 *For any quantified Horn formula* $\Phi \in QHORN^*$ *with free variables, there exists an equivalent formula* $\Phi' \in \exists HORN^*$ *without universal quantifiers. The length of* Φ' *is bounded by* $|\forall| \cdot |\Phi|$*, where* $|\forall|$ *is the number of universal quantifiers in* Φ*, and* $|\Phi|$ *is the length of* Φ*.*

Let $\Phi \in QHORN^*$ with $\Phi(\mathbf{z}) = \forall X_1 \exists Y_1 ... \forall X_r \exists Y_r \, \phi(X_1, ..., X_r, Y_1, ..., Y_r, \mathbf{z})$ where $X_i = (x_{i,1}, ..., x_{i,n_i})$ and $Y_i = (y_{i,1}, ..., y_{i,m_i})$ ($n_i \geq 1$ and $m_i \geq 1$, $i = 1, ..., r$, $r \geq 1$), be a quantified Horn formula whose outermost quantifiers are universal and whose innermost quantifiers are existential.

Listing 1 presents an algorithm to transform Φ into $\Phi_{\exists\text{poly}}$ as described above.

Listing 1. The Transformation Algorithm

```
// Input: Φ ∈ QHORN*, Φ(z) = ∀X_1∃Y_1...∀X_r∃Y_r φ(X_1,...,X_r,Y_1,...,Y_r,z),
//          where X_i = (x_{i,1},...,x_{i,n_i}) and Y_i = (y_{i,1},...,y_{i,m_i})
// Output: The matrix of Φ_∃poly ∈ ∃HORN* with Φ_∃poly ≈ Φ

φ_∃poly = ∅;
for (i = 1 to r) do
  for (j = 1 to n_i) do A_{x_{i,j}} = 1;
for (i = 1 to r) do {
  for (j = 1 to n_i) do {
    A_{x_{i,j}} = 0;
    for (k = i to r) do
      for (l = 1 to m_k) do y'_{k,l} = new ∃-var;
    φ_∃poly = φ_∃poly ∪ φ[x/A_x, y/y']; // (*)
    A_{x_{i,j}} = 1;
  }
  for (l = 1 to m_i) do y'_{i,l} = new ∃-var;
}
φ_∃poly = φ_∃poly ∪ φ[x/A_x, y/y']; // (*)
```

In the main loop of the algorithm, one universal variable $x_{i,j}$ is given the value false, while all the others are true. For any such assignment $A_{\mathbf{x}}$, all existential variables which are dominated by $x_{i,j}$ (i.e. their corresponding quantifier follows $\forall x_{i,j}$) have to be replaced by independent new variables $\mathbf{y}'$. Then, the matrix of the original formula has to be duplicated, with $A_{\mathbf{x}}$ being substituted for $\mathbf{x}$ and $\mathbf{y}'$ being substituted for $\mathbf{y}$. After executing the main loop, one additional copy is needed for the case where all universal variables are true. Notice that we treat the existential variables as objects. If we let $y'_{i,j} =$ `new ∃-var` and use this variable in multiple locations, then all share the same variable object, which means all those subformulas share that existential variable.

The lines marked with (*) need time $O(|\Phi|)$. They are executed $n_1 + ... + n_r + 1 = |\forall| + 1$ times, so the algorithm in total requires time $O(|\forall| \cdot |\Phi|)$.

4 Satisfiability Testing for $QHORN^*$ Formulas

Let $\Phi(\mathbf{z}) \in \exists HORN^*$ be an existentially quantified Horn formula of the form $\Phi(\mathbf{z}) = \exists y_1 ... \exists y_m \, \phi(y_1, ..., y_m, \mathbf{z})$. Then $\Phi(\mathbf{z})$ is satisfiable if and only if its matrix $\phi(y_1, ..., y_m, \mathbf{z})$ is satisfiable. The latter is a purely propositional formula, therefore a SAT solver for propositional Horn formulas can be used to determine the satisfiability of an arbitrary formula in $\exists HORN^*$. That makes $\exists HORN^*$ a suitable representation for satisfiability testing. Moreover, as we have just shown in Section 3, we can efficiently transform arbitrary $QHORN^*$ formulas into this special form. These observations suggest that we should always take this route. We then obtain the following algorithm for determining the satisfiability of a formula $\Psi \in QHORN^*$:

(1) Transform Ψ into $\Psi_{\exists\text{poly}} \in \exists HORN^*$ with $|\Psi_{\exists\text{poly}}| = O(|\forall| \cdot |\Psi|)$. This requires time $O(|\forall| \cdot |\Psi|)$ as discussed in Section 3.4.
(2) Determine the satisfiability of $\psi_{\exists\text{poly}}$, which is the purely propositional matrix of $\Psi_{\exists\text{poly}}$. It is well known (see [13]) that SAT for propositional Horn formulas can be solved in linear time, here $O(|\psi_{\exists\text{poly}}|) = O(|\forall| \cdot |\Psi|)$.

In total, the algorithm requires time $O(|\forall| \cdot |\Psi|)$. The best existing algorithm presented in [9] has the same complexity, but that algorithm is significantly more complicated and cannot directly reuse existing propositional SAT solvers like this new algorithm does.

5 Satisfiability Models for $QHORN$ Formulas

The findings on partial satisfiability models in Section 3.2 have enabled us to transform arbitrary quantified Horn formulas into a very simple structure as shown above. But besides this main result, the work on partial satisfiability models can also provide us with more insight into the structure of (total) satisfiability models for quantified Horn formulas without free variables. That enables us to better understand the general behavior of the quantified variables. Moreover, this section will outline an efficient algorithm for finding (total) satisfiability models for $QHORN$ formulas.

5.1 Structure of the Models

We start with showing that satisfiable $QHORN$ formulas have models consisting of functions of the form $f_y(x_1, ..., x_n) = \bigwedge_{i \in I} x_i$ (or the constants $f_y = 0$

resp. $f_y = 1$). In accordance with [8], this class of models is called K_2.

Definition 12 *Let*

$$\begin{aligned} K_2 := \quad & \{f \mid \exists I \subseteq \{1,...,n\} : f(x_1,...,x_n) = \textstyle\bigwedge_{i\in I} x_i, n \geq 1\} \\ & \cup \{f \mid f = 0 \text{ or } f = 1\} \end{aligned}$$

be a class of Boolean functions, and let $M = (f_{y_1},...,f_{y_m})$ be a satisfiability model for a formula $\Phi \in QBF$. Then we call M a $\mathbf{K_2}$ ***satisfiability model*** *for Φ if the model functions f_{y_i} are in K_2 for every $1 \leq i \leq m$.*

Theorem 13 *Any satisfiable QHORN formula has a K_2 satisfiability model.*

Proof:

If Φ is satisfiable, it has a $Z_{\leq 1}$-partial satisfiability model $M = (f_{y_1},...,f_{y_m})$. According to Definition 5 and Theorem 7, its total completion M^t is a (total) satisfiability model and is composed of functions given by

$$\begin{aligned} f^t_{y_i}(x_1,...,x_{n_i}) := \quad & (x_1 \ \vee f_{y_i}(0,1,1,...,1)) \\ \wedge \ & (x_2 \ \vee f_{y_i}(1,0,1,...,1)) \\ \wedge \ & ... \\ \wedge \ & (x_{n_i} \vee f_{y_i}(1,1,...,1,0)) \\ \wedge \ & f_{y_i}(1,...,1) \end{aligned}$$

Notice that $f_{y_i}(0,1,1,...,1)$, $f_{y_i}(1,0,1,...,1)$, ..., $f_{y_i}(1,...,1)$ are merely Boolean constants in the definition of $f^t_{y_i}$. So we actually have functions of the form

$$\begin{aligned} f^t_{y_i}(x_1,...,x_{n_i}) := \quad & (x_1 \ \vee c_1) \\ \wedge \ & (x_2 \ \vee c_2) \\ \wedge \ & ... \\ \wedge \ & (x_{n_i} \vee c_{n_i}) \\ \wedge \ & c_{n_i+1} \end{aligned}$$

with $c_j = 0$ or $c_j = 1$. Clearly, those functions are in K_2. $\square$

In [8], it has already been shown that quantified Horn formulas have K_2 models. However, that proof was significantly longer and required more advanced techniques (Q-pos-unit-resolution). Most importantly, however, it did not lead to an efficient algorithm for finding those K_2 models. It has since been an open question whether it would be possible to find K_2 satisfiability models in time at most $O(|\forall|\cdot|\Phi|)$, the complexity of $QHORN$-SAT (see Section 4). As shown in the following section, the new approach with partial satisfiability models also solves this problem and provides an $O(|\forall| \cdot |\Phi|)$-algorithm.

5.2 Finding Models

The algorithm for finding satisfiability models is actually a byproduct of the quantifier elimination in Section 3.3: the proof of Lemma 9 describes how a $Z_{\leq 1}$-partial satisfiability model for a formula $\Phi \in QHORN$ is obtained by solving $\Phi_{\exists\text{poly}}$. This leads to the following basic algorithm for finding a K_2 satisfiability model for Φ:

(1) Transform Φ into $\Phi_{\exists\text{poly}}$ and solve it.
(2) Obtain a $Z_{\leq 1}$-partial satisfiability model as in the proof of Lemma 9.
(3) Use total completion (Definition 5) to build a total satisfiability model.

A closer look at steps 2 and 3 shows that we do not actually have to write down the $Z_{\leq 1}$-partial satisfiability model, because the total expansion only needs certain values of the partial model:

$$\begin{aligned} f^t_{y_i}(x_1, ..., x_{n_i}) := \quad & (x_1 \;\;\, \vee f_{y_i}(0, 1, 1, ..., 1)) \\ & \wedge\, (x_2 \;\;\, \vee f_{y_i}(1, 0, 1, ..., 1)) \\ & \wedge\, ... \\ & \wedge\, (x_{n_i} \vee f_{y_i}(1, 1, ..., 1, 0)) \\ & \wedge\, f_{y_i}(1, ..., 1) \end{aligned}$$

In this excerpt from Definition 5, $f^t_{y_i}$ is the total expansion, and f_{y_i} belongs to the partial model. According to the proof of Lemma 9, the f_{y_i} are given as

$$\begin{aligned} f_{y_i}(x_1, ..., x_{n_i}) = \quad & (\bar{x}_1 \wedge x_2 \wedge ... \wedge x_{n_i} \quad \to t(y_i^{(0,1,...,1)})) \\ & \wedge\, (x_1 \wedge \bar{x}_2 \wedge x_3 \wedge ... \wedge x_{n_i} \quad \to t(y_i^{(1,0,1,...,1)})) \\ & \wedge\, ... \\ & \wedge\, (x_1 \wedge ... \wedge x_{n_i - 1} \wedge \bar{x}_{n_i} \quad \to t(y_i^{(1,...,1,0)})) \\ & \wedge\, (x_1 \wedge ... \wedge x_{n_i} \quad \to t(y_i^{(1,...,1)})) \end{aligned}$$

where t is a satisfying truth assignment to the existentials y_i^A (the copies of y_i) in $\Phi_{\exists \mathrm{poly}}$. Now, notice that $f_{y_i}(0, 1, ..., 1) = t(y_i^{(0,1,...,1)})$, etc. That allows us to combine both definitions, and we obtain the following theorem.

Theorem 14 *Let $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in QHORN$ be a quantified Horn formula with universal variables $\mathbf{x} = (x_1, ..., x_n)$ and existentials $\mathbf{y} = (y_1, ..., y_m)$. We require that Φ is satisfiable, which means its expansion $\Phi_{\exists \mathrm{poly}}$ is also satisfiable. Let t be a satisfying truth assignment to the existentials in $\Phi_{\exists \mathrm{poly}}$. Then $M = (f_{y_1}, .., f_{y_m})$ with*

$$\begin{aligned} f_{y_i}(x_1, ..., x_{n_i}) := \quad & (x_1 \quad \vee t(y_i^{(0,1,1,...,1)})) \\ & \wedge (x_2 \quad \vee t(y_i^{(1,0,1,...,1)})) \\ & \wedge \ldots \\ & \wedge (x_{n_i} \vee t(y_i^{(1,1,...,1,0)})) \\ & \wedge t(y_i^{(1,...,1)}) \end{aligned}$$

is a satisfiability model for Φ.

This allows us to refine the algorithm for finding a K_2 satisfiability model for a formula $\Phi \in QHORN$:

(1) Transform Φ into $\Phi_{\exists \mathrm{poly}}$ and solve it. If $\Phi_{\exists \mathrm{poly}}$ is unsatisfiable, Φ has no satisfiability model. Otherwise, we obtain a satisfying truth assignment to the existentials in $\Phi_{\exists \mathrm{poly}}$.
(2) Use this assignment to construct a K_2 satisfiability model as described in Theorem 14.

The first step requires time $O(|\forall| \cdot |\Phi|)$ (see Section 4), and the second needs time $O(|\exists| \cdot |\forall|)$. In total, we can find the model in $O(|\forall| \cdot |\Phi| + |\exists| \cdot |\forall|) = O(|\forall| \cdot |\Phi|)$.

Corollary 15 *Let $\Phi \in QHORN \cap QSAT$ be a satisfiable quantified Horn formula. Then we can find a K_2 satisfiability model for Φ in time $O(|\forall| \cdot |\Phi|)$, where $|\forall|$ is the number of universal quantifiers in Φ and $|\Phi|$ the length of Φ.*

6 Equivalence Models for $QHORN^*$ Formulas

As pointed out earlier (in Section 3.2), satisfiability models are only defined for closed formulas, i.e. for formulas without free variables. This restriction can often be circumvented by considering fixed assignments to the free variables and then treating the formula with fixed free variables as a closed formula. This trick allowed us to establish many results of this paper for formulas with free variables, too. Nevertheless, it would be useful to have "native" models for QBF^* and generalize the results for satisfiability models to these models.

As introduced in [14], *equivalence models* extend the notion of models to formulas with free variables by allowing that the propositional formulas f_{y_i} may also contain free variables. Instead of requiring that $\Phi[\mathbf{y}/M]$ must be satisfiable, equivalence models demand that Φ and $\Phi[\mathbf{y}/M]$ must be equivalent. That makes the concept fit nicely with the main application of QBF^* formulas, which is to provide an equivalent (potentially shorter) representation of propositional formulas. Formally, equivalence models are defined as follows:

Definition 16 *Let $\Phi(\mathbf{z}) = Q\,\phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be a quantified Boolean formula with prefix Q and matrix ϕ, universal variables $\mathbf{x} = (x_1, ..., x_n)$, existential variables $\mathbf{y} = (y_1, ..., y_m)$ and free variables $\mathbf{z} = (z_1, ..., z_r)$. For propositional formulas f_{y_i} over $\mathbf{z}$ and over universal variables whose quantifiers precede $\exists y_i$, we say $M = (f_{y_1}, ..., f_{y_m})$ is an **equivalence model** for $\Phi(\mathbf{z})$ if and only if $\Phi(\mathbf{z}) \approx \forall x_1 ... \forall x_n\, \phi(x_1, ..., x_n, \mathbf{y}, \mathbf{z})[\mathbf{y}/M]$.*

We have shown that closed quantified Horn formulas have K_2 satisfiability models, which means the model functions are conjunctions of positive universal variables. The question is how this generalizes to equivalence models for $QHORN^*$ formulas. We managed to come up with the following answer: the model functions are now conjunctions and disjunctions of positive universals and free variables. Thus it seems that the absence of negation in the model functions is a characteristic feature of quantified Horn formulas which is still preserved when free variables are allowed.

More formally, we have been able to prove that quantified Horn formulas have monotone equivalence models. We first define what monotony means here.

Definition 17 *Let $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{x}' = (x'_1, .., x'_n) \in \{0, 1\}^n$, and let $f : \{0, 1\}^n \to \{0, 1\}$ be a Boolean function. Then f is **monotone** if and only if $\mathbf{x} \leq \mathbf{x}'$ implies $f(\mathbf{x}) \leq f(\mathbf{x}')$, with canonical ordering $0 \leq 1$ and $\mathbf{x} \leq \mathbf{x}'$ iff $x_i \leq x'_i$ for all i.*

We usually represent the Boolean functions from which equivalence models are composed as propositional formulas. This leads to an equivalent characterization of monotony:

Proposition 18 *(based on [15])*

A Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ is monotone if and only if it can be represented as a propositional formula F which contains only positive literals and the reduced operator set $\{\wedge, \vee\}$. We also allow $F = 0$ resp. $F = 1$.

In the following discussion, we always use this characterization of monotony.

Definition 19 *Let $M = (f_{y_1}, ..., f_{y_m})$ be an equivalence model for a quantified Boolean formula $\Phi \in QBF^*$. Then M is a* ***monotone equivalence model*** *if and only if the functions f_{y_i}, $1 \leq i \leq m$, are monotone.*

Notice that when we substitute an arbitrary monotone model M for the existential variables, the formula $\Phi[\mathbf{y}/M]$ may not be in *CNF* anymore. Of course, it can be transformed into *CNF* with the laws of associativity and distributivity and DeMorgan's laws, but another problem may then occur: the resulting *CNF* formula is not necessarily a Horn formula.

In our proof, however, the construction of the model assures that $\Phi[\mathbf{y}/M]$ is a quantified Horn formula when transformed into *CNF*. The class of non-*CNF* formulas that may be transformed into *CNF* formulas with the Horn property shall be denoted with $QHORN_L^*$ as defined below.

Definition 20 *With $QHORN_L^*$, we denote the class of quantified Boolean formulas $\Phi \in QBF^*$ for which there exist $\Phi' \in QHORN^*$ such that Φ' can be obtained from Φ by applying the laws of associativity and distributivity and DeMorgan's laws.*

We now show that a quantified Horn formula always has a monotone equivalence model. In the proof, we inductively construct such a model for any $\Phi \in QHORN^*$.

Theorem 21 *Any formula $\Phi \in QHORN^*$ has a monotone equivalence model $M = (f_{y_1}, ..., f_{y_m})$. Moreover, M can be chosen such that $\Phi[\mathbf{y}/M] \in QHORN_L^*$.*

Proof: If $\Phi(\mathbf{z})$ is unsatisfiable, there is a $\{0, 1\}$-equivalence model, and therefore a monotone equivalence model M with $\Phi[\mathbf{y}/M] \in QHORN_L^*$. For the remainder of this proof, we assume the satisfiability of the input formula and prove the theorem by induction on the number of quantifiers.

For $k = 1$, we have a formula with one quantifier, which may be universal or existential. If $\Phi(\mathbf{z}) = \forall x_1\, \phi(x_1, \mathbf{z})$ with a propositional formula ϕ, then the empty model $M = ()$ is a monotone equivalence model for Φ.

The second case in which the quantifier is existential is more interesting. Suppose Φ is given as $\Phi(\mathbf{z}) = \exists y_1\, \phi(y_1, \mathbf{z})$ with a propositional formula ϕ. If y_1 or $\neg y_1$ occurs in $\phi(y_1, \mathbf{z})$ as a unit clause, define $f_{y_1} = 1$ or $f_{y_1} = 0$, respectively. If y_1 occurs only positively or only negatively in $\phi(y_1, \mathbf{z})$, let $f_{y_1} = 1$ or $f_{y_1} = 0$. Otherwise, if y_1 occurs both positively and negatively, let $\neg a_{i,1} \vee ... \vee \neg a_{i,s_i} \vee y_1 := \neg A_i \vee y_1$ be the clauses in which y_1 occurs positively ($1 \leq i \leq c_{pos}$, where c_{pos} is the number of those clauses). Analogously, let $b_{j,1} \vee ... \vee b_{j,t_j} \vee \neg y_1 := B_j \vee \neg y_1$ be the clauses in which y_1 occurs negatively

$(1 \leq j \leq c_{neg})$. Finally, let C be the clauses which contain neither y_1 nor $\neg y_1$. Clauses which contain both y_1 and $\neg y_1$ are tautological and can therefore be removed from the formula. If y_1 only occurs in tautological clauses, we can also remove that variable itself.
We now define the model of y_1. The idea is to choose a model such that tautological clauses are created when f_{y_1} is substituted for positive instances of y_1, while substituting f_{y_1} for the negative instances of y_1 produces the expansion $\phi(0, \mathbf{z}) \vee \phi(1, \mathbf{z})$ of the existentially quantified formula $\exists y_1\, \phi(y_1, \mathbf{z})$. That can be accomplished with the following definition:

$$f_{y_1} = \bigvee_{1 \leq i \leq c_{pos}} A_i = \bigvee_{1 \leq i \leq c_{pos}} (a_{i,1} \wedge \ldots \wedge a_{i,s_i})$$

For a clause $\neg A_i \vee y_1$ in which y_1 occurs positively, we obtain $\neg A_i \vee f_{y_1} = \neg A_i \vee A_1 \vee \ldots \vee A_i \vee \ldots \vee A_{c_{pos}}$, which contains both A_i and $\neg A_i$ and is thus tautological.
On the other hand, consider the set of clauses in which y_1 occurs negatively:

$$\left(\bigwedge_j (B_j \vee \neg y_1)\right)[y_1/f_{y_1}] = \bigwedge_j \left(B_j \vee \neg \left(\bigvee_i A_i\right)\right)$$

$$\approx \bigwedge_j \left(B_j \vee \left(\bigwedge_i \neg A_i\right)\right) \approx \bigwedge_{i,j} (\neg A_i \vee B_j)$$

The clauses C which do not contain y_1 (respectively $\neg y_1$) remain unchanged. As motivated before, the resulting formula $\Phi[y_1/f_{y_1}] \approx \bigwedge_{i,j} (\neg A_i \vee B_j) \wedge C$ is the expansion of the existentially quantified formula $\exists y_1\, \phi(y_1, \mathbf{z})$, which can be seen as follows:

$$\begin{aligned}
\exists y_1\, \phi(y_1, \mathbf{z}) &\approx \phi(0, \mathbf{z}) \vee \phi(1, \mathbf{z}) \\
&\approx \left(\bigwedge_i \neg A_i \wedge C\right) \vee \left(\bigwedge_j B_j \wedge C\right) \\
&\approx \left(\left(\bigwedge_i \neg A_i\right) \vee \left(\bigwedge_j B_j\right)\right) \wedge C \\
&\approx \bigwedge_{i,j} (\neg A_i \vee B_j) \wedge C
\end{aligned}$$

This proves that $M = (f_{y_1})$ is an equivalence model for $\Phi(\mathbf{z})$. Notice that $\bigwedge_{i,j} (\neg A_i \vee B_j) \wedge C$ is a Horn formula, because the $\neg A_i$ contain only negative literals, and each B_j has at most one positive literal. Thus, $\Phi[y_1/f_{y_1}] \in QHORN^*_L$.

Now let $k > 1$. Again, we have two cases: the outer quantifier may either be universal or existential. If it is universal, Φ has the form $\Phi(\mathbf{z}) = \forall x_k\, \Phi'(x_k, \mathbf{z})$, where Φ' is a formula with $k-1$ quantifiers. If $\Phi \in QHORN^*$, then also $\Phi' \in QHORN^*$, and by the induction hypothesis, Φ' has a monotone equivalence model $M_{\Phi'}$ with $\Phi'[\mathbf{y}/M_{\Phi'}] \in QHORN_L^*$. $M_{\Phi'}$ is also a monotone equivalence model for Φ, because $\Phi' \approx \Phi'[\mathbf{y}/M_{\Phi'}]$ implies

$$\begin{aligned}\Phi(\mathbf{z}) \;=\; \forall x_k\, \Phi'(x_k,\mathbf{z}) \;\approx\; \forall x_k\, (\Phi'(x_k,\mathbf{z})[\mathbf{y}/M_{\Phi'}]) &= (\forall x_k\, \Phi'(x_k,\mathbf{z}))[\mathbf{y}/M_{\Phi'}] \\ &= \Phi(\mathbf{z})[\mathbf{y}/M_{\Phi'}]\end{aligned}$$

Obviously, $\Phi[\mathbf{y}/M_{\Phi'}] \in QHORN_L^*$ as well.

In the second case, the outer quantifier is existential, and Φ has the form $\Phi(\mathbf{z}) = \exists y_k\, \Phi'(y_k,\mathbf{z})$. Notice that y_k is a free variable in Φ'. If Φ' contains only universal quantifiers, we can remove all of them, as they do not dominate any existentially quantified variables. We are then left with only one existential variable and can proceed as in the induction base. For the remainder of this proof, we assume that Φ' contains at least one existentially quantified variable. As above, Φ' is a formula with $k-1$ quantifiers, and according to the induction hypothesis, it has a monotone equivalence model $M_{\Phi'} = (f'_{y_1},\dots,f'_{y_{k-1}})$ with $\Phi'[\mathbf{y}/M_{\Phi'}] \in QHORN_L^*$. $\Phi'(y_k,\mathbf{z}) \approx \Phi'(y_k,\mathbf{z})[\mathbf{y}/M_{\Phi'}]$ implies

$$\Phi(\mathbf{z}) = \exists y_k \Phi'(y_k,\mathbf{z}) \approx \exists y_k\, (\Phi'(y_k,\mathbf{z})[\mathbf{y}/M_{\Phi'}])$$

$\Phi'[\mathbf{y}/M_{\Phi'}] \in QHORN_L^*$ means that there exists $\Phi''(\mathbf{z}) \in QHORN^*$ with $\Phi''(\mathbf{z}) \approx \exists y_k\, (\Phi'(y_k,\mathbf{z})[\mathbf{y}/M_{\Phi'}])$. Under the assumption that Φ' contains at least one existential variable, $\Phi'[\mathbf{y}/M_{\Phi'}]$ has less than $k-1$ quantifiers. Thus, Φ'' has less than k quantifiers, and only the outermost is existential. By the induction hypothesis, it has a monotone equivalence model $M_{\Phi''} = (f''_{y_k})$ with $\Phi''[y_k/f''_{y_k}] \in QHORN_L^*$.
We now combine $M_{\Phi'} = (f'_{y_1},\dots,f'_{y_{k-1}})$ and $M_{\Phi''} = (f''_{y_k})$ into a monotone equivalence model $M = (f_{y_1},\dots,f_{y_k})$ for the original formula Φ by assigning $f_{y_i} = f'_{y_i}[y_k/f''_{y_k}]$ for $1 \le i \le k-1$ and $f_{y_k} = f''_{y_k}$. It is obvious that M is monotone. Informally, it is also clear that M is an equivalence model for Φ, but the formal proof is somewhat tedious:

$$\begin{aligned}&\Phi(\mathbf{z})\\ \approx\;& \Phi''(\mathbf{z})\\ \approx\;& \Phi''(\mathbf{z})[y_k/f''_{y_k}]\\ \approx\;& (\exists y_k\, (\Phi'(y_k,\mathbf{z})[y_1/f'_{y_1},\dots,y_{k-1}/f'_{y_{k-1}}]))[y_k/f''_{y_k}]\\ =\;& (\exists y_k\, \Phi'(y_k,\mathbf{z}))[y_1/f'_{y_1}[y_k/f''_{y_k}],\dots,y_{k-1}/f_{y_{k-1}}[y_k/f''_{y_k}],y_k/f''_{y_k}]\\ =\;& (\exists y_k\, \Phi'(y_k,\mathbf{z}))[y_1/f_{y_1},\dots,y_k/f_{y_k}]\\ =\;& \Phi(\mathbf{z})[\mathbf{y}/M]\end{aligned}$$

$\Phi(\mathbf{z})[\mathbf{y}/M] \in QHORN^*_L$, because $\Phi''(\mathbf{z})[y_k/f''_{y_k}] \in QHORN^*_L$ and $\Phi(\mathbf{z})[\mathbf{y}/M] \approx \Phi''(\mathbf{z})[y_k/f''_{y_k}]$. $\square$

The previous result reveals the structure of equivalence models for $QHORN^*$ formulas. Unfortunately, the proof itself does not lead to a feasible algorithm for finding those equivalence models. The problem with the algorithm suggested by the proof is that the formula which is being worked on may blow up exponentially. As that algorithm moves step by step from the innermost quantifiers to the outermost quantifiers, the model found in the previous step is always substituted into the given formula which is then re-transformed into *CNF*. For certain formulas (see [9]), this may cause exponential growth. Further research should investigate whether there exist better algorithms for finding equivalence models for $QHORN^*$. It is also unclear whether the relationship between partial and total satisfiability models for closed formulas has a counterpart for equivalence models.

7 Conclusions

This paper has demonstrated that the syntactic restriction of allowing at most one positive literal per clause influences the semantics of quantified Horn formulas with an interesting effect on the behavior of the quantifiers. We have shown that only cases where at most one of the universally quantified variables is false are relevant for the choice of the existential variables. This has allowed us to provide a detailed characterization of satisfiability models for $QHORN$ formulas by focusing only on the relevant parts of the model. Accordingly, the concept of $R_\forall$-partial satisfiability models has been introduced, and it has been shown that for $QHORN$ formulas, the partial model can always be extended to a total satisfiability model.

Based on these results, we have been able to show that

- any formula $\Phi \in QHORN^*$ of length $|\Phi|$ with free variables, $|\forall|$ universal quantifiers and an arbitrary number of existential quantifiers can be transformed into an equivalent quantified Horn formula of length $O(|\forall| \cdot |\Phi|)$ which contains only existential quantifiers.
- $QHORN^*$-SAT can be solved in time $O(|\forall| \cdot |\Phi|)$ by transforming the input formula into a satisfiability-equivalent propositional formula.
- satisfiability models for $QHORN$ formulas can be found in time $O(|\forall| \cdot |\Phi|)$.

We have also investigated models for $QHORN^*$ formulas with free variables and have proved that these equivalence models are monotone.

Further research should continue investigating equivalence models, because compared to the wealth of results on satisfiability models for closed formulas, our understanding of equivalence models is still rather limited. In particular, it must be investigated how to efficiently compute them for given formulas. In addition, it might be interesting to conduct experimental studies on the structure of satisfiability/equivalence models for different instances of quantified Horn formulas (i.e. random formulas, formulas with a special structure, etc.).

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