Dependency Quantified Horn Formulas: Models and Complexity

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Abstract. Dependency quantified Boolean formulas (*DQBF*) extend quantified Boolean formulas with Henkin-style partially ordered quantifiers. It has been shown that this is likely to yield more succinct representations at the price of a computational blow-up from *PSPACE* to *NEXPTIME*. In this paper, we consider dependency quantified Horn formulas (*DQHORN*), a subclass of *DQBF*, and show that the computational simplicity of quantified Horn formulas is preserved when adding partially ordered quantifiers.

We investigate the structure of satisfiability models for *DQHORN* formulas and prove that for both *DQHORN* and ordinary *QHORN* formulas, the behavior of the existential quantifiers depends only on the cases where at most one of the universally quantified variables is zero. This allows us to transform *DQHORN* formulas with free variables into equivalent *QHORN* formulas with only a quadratic increase in length.

An application of these findings is to determine the satisfiability of a dependency quantified Horn formula Φ with $|\forall|$ universal quantifiers in time $O(|\forall| \cdot |\Phi|)$, which is just as hard as *QHORN*-SAT.

1 Introduction

The language of Quantified Boolean Formulas (*QBF*) offers a concise way to represent formulas which arise in areas such as planning, scheduling or verification [15, 17]. *QBF* formulas are usually assumed by definition to be in prenex form such that all quantifiers appear at the beginning, and that is also the input format generally required by *QBF* solvers. This does, however, impose a total ordering on the quantifiers where each existentially quantified variable depends on all preceding universal variables. Consider the following example:

$$\forall x_1 \left[(\forall x_2 \exists y_1 \phi(x_1, x_2, y_1)) \wedge (\forall x_3 \exists y_2 \psi(x_1, x_3, y_2)) \right]$$

In this non-prenex formula, the choice for y_1 depends on the values of x_1 and x_2, and y_2 depends on x_1 and x_3. Using quantifier rewriting rules, we can obtain the equivalent prenex formula

$$\forall x_1 \forall x_2 \exists y_1 \forall x_3 \exists y_2 \; \phi(x_1, x_2, y_1) \wedge \psi(x_1, x_3, y_2)$$

As above, y_1 depends on x_1 and x_2, but y_2 now depends on x_1, x_2 and x_3, so we lose some of the structural information inherent in the original formula.

Recent experimental studies [7, 8] have shown that this problem may have a considerable impact on the performance of *QBF* solvers. Accordingly, different solutions have lately been suggested to overcome the problem, e.g. by recovering lost information from the formula structure (in particular from the local connectivity of variables in common clauses) [2, 3], or by extending *QBF* solvers to directly handle non-prenex formulas [8].

Another solution has been proposed by Henkin [9] for first-order predicate logic. He has introduced partially ordered quantifiers, called *branching quantifiers* or simply *Henkin quantifiers*, as in the expression

$$\begin{pmatrix} \forall x_1 \forall x_2 \exists y_1 \\ \forall x_1 \forall x_3 \exists y_2 \end{pmatrix} \; \phi(x_1, x_2, y_1) \wedge \psi(x_1, x_3, y_2)$$

which correctly preserves the dependencies from our introductory example. Since the only relevant information is which universal quantifiers precede which existential quantifier, we can use a (typographically) simpler function-like notation as follows:

$$\forall x_1 \forall x_2 \exists y_1(x_1, x_2) \forall x_3 \exists y_2(x_1, x_3) \; \phi(x_1, x_2, y_1) \wedge \psi(x_1, x_3, y_2)$$

For each existential quantifier, we indicate the universal variables on which it depends. Without loss of information, we can assume that the prefix is in the form $\forall^* \exists^*$:

$$\forall x_1 \forall x_2 \forall x_3 \exists y_1(x_1, x_2) \exists y_2(x_1, x_3) \; \phi(x_1, x_2, y_1) \wedge \psi(x_1, x_3, y_2)$$

This notation has been introduced for quantified Boolean formulas by Peterson, Azhar and Reif [16] under the name *Dependency Quantified Boolean Formulas (DQBF)*.

Notice that partially ordered quantifiers do not only eliminate the aforementioned loss of information due to prenexing, caused by flattening a tree-like hierarchy of quantifiers and corresponding scopes into a linear ordering. The Henkin approach is significantly more general than the suggestions above, because it allows to express subtle dependencies where the hierarchy of quantifier scopes is no longer tree-like. For example, we could add an existential variable y_3 to our sample formula, such that y_3 depends on x_2 and x_3 as indicated in the following prefix:

$$\forall x_1 \forall x_2 \forall x_3 \exists y_1(x_1, x_2) \exists y_2(x_1, x_3) \exists y_3(x_2, x_3)$$

It is not clear how this prefix could be represented in a succinct QBF, even if we allow non-prenex formulas.

Partially-ordered quantification has been around for quite some time, but has not been widely used in combination with quantified Boolean formulas. This is probably due to the fact that *DQBF* is NEXPTIME-complete, as has been shown by Peterson, Azhar and Reif [16]. Assuming that NEXPTIME$\neq$PSPACE, this means that there are *DQBF* formulas for which no equivalent *QBF* of polynomial length can be computed in polynomial space. It also means a jump in complexity compared to *QBF* which is PSPACE-complete. The latter is already considered quite hard, but continued research and the lifting of propositional SAT techniques to *QBF*s have recently produced interesting improvements (see, e.g., [3, 14, 18]) and have led to the emergence of more powerful *QBF*-SAT solvers [13]. In addition, tractable subclasses of *QBF* have been identified and investigated, e.g. *QHORN*, which contains all *QBF* formulas in conjunctive normal form (*CNF*) whose clauses have at most one positive literal. This subclass is important, because it is sufficient for expressing simple "`if-then`" statements, and because *QHORN* formulas may occur as subproblems when solving arbitrary *QBF* formulas [5].

In this paper, we consider dependency quantified Horn formulas (*DQHORN*), the dependency quantified equivalent to *QHORN*. Our main contribution is to prove that *DQHORN* is a tractable subclass of *DQBF* and is in fact just as difficult as *QHORN*. To be more precise, we present an algorithm which can determine the satisfiability of a *DQHORN* formula Φ with free variables, $|\forall|$ universal quantifiers and an arbitrary number of existential quantifiers in time $O(|\forall| \cdot |\Phi|)$.

We achieve this by investigating the interplay of existential and universal quantifiers with the help of satisfiability models. This concept has been introduced in [12], and Section 3 shows how it can be extended for *DQBF* formulas. We prove that for both *DQHORN* and ordinary *QHORN*, the behavior of the existential quantifiers depends only on the cases where at most one of the universally quantified variables is zero. In Section 4, we demonstrate how *DQBF* formulas with free variables can be transformed into equivalent *QBF* formulas by expanding the universal quantifiers. This expansion may cause an exponential blowup for arbitrary formulas. But the results from Section 3 allow us to avoid this for *DQHORN* formulas with free variables by applying a generalization of the special expansion method that we have presented in [4] for *QHORN*. Finally, an algorithm for solving *DQHORN*-SAT is developed in Section 5.

2 Preliminaries

In this section, we recall the basic terminology and notation for *QBF* and introduce *DQBF*.

A quantified Boolean formula $\Phi \in QBF$ in prenex form is a formula

$$\Phi = Q_1 v_1 ... Q_k v_k \, \phi(v_1, ..., v_k)$$

with quantifiers $Q_i \in \{\forall, \exists\}$ and a propositional formula $\phi(v_1, ..., v_k)$ over variables $v_1, ..., v_k$. We call $Q := Q_1 v_1 ... Q_k v_k$ the *prefix* and ϕ the *matrix* of Φ. Variables which are bound by universal quantifiers are called *universal variables* and are usually given the names $x_1, ..., x_n$. Analogously, variables in the scope of an existential quantifier are *existential variables* and have names $y_1, ..., y_m$. We write $\Phi = Q\ \phi(\mathbf{x}, \mathbf{y})$ or simply $\Phi = Q\ \phi$.

Variables which are not bound by quantifiers are *free variables.* Formulas without free variables are said to be *closed.* If free variables are allowed, we indicate this with an additional star * after the name of the formula class. Accordingly, *QBF* is the class of closed quantified Boolean formulas, and QBF^* denotes the quantified Boolean formulas with free variables. We write $\Phi(\mathbf{z}) = Q\ \phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$ or $\Phi(\mathbf{z}) = Q\ \phi(\mathbf{z})$ for a QBF^* formula with free variables $\mathbf{z} = (z_1, ..., z_r)$. A closed *QBF* formula is either true or false, whereas the truth value of a QBF^* formula depends on the value of the free variables. Two QBF^* formulas $\Psi_1(z_1, ..., z_r)$ and $\Psi_2(z_1, ..., z_r)$ are said to be equivalent ($\Psi_1 \approx \Psi_2$) if and only if $\Psi_1 \models \Psi_2$ and $\Psi_2 \models \Psi_1$, where semantic entailment $\models$ is defined as follows: $\Psi_1 \models \Psi_2$ if and only if for all truth assignments $t(\mathbf{z}) := (t(z_1), ..., t(z_r)) \in \{0, 1\}^r$ to the free variables $\mathbf{z} = (z_1, ..., z_r)$, we have $\Psi_1(t(\mathbf{z})) = 1 \Rightarrow \Psi_2(t(\mathbf{z})) = 1$.

For *DQBF* formulas, we introduce a notation which allows us to quickly enumerate the dependencies for a given existential variable y_i ($1 \leq i \leq m$). We are using indices $d_{i,1}, ..., d_{i,n_i}$ which point to the n_i universals on which y_i depends. For example, given the existential quantifier $\exists y_4(x_3, x_5)$, we say that y_4 depends on $x_{d_{4,1}}$ and $x_{d_{4,2}}$ with $d_{4,1} = 3$ and $d_{4,2} = 5$.

With this notation, a dependency quantified Boolean formula $\Phi \in DQBF$ with universal variables $\mathbf{x} = (x_1, ..., x_n)$ and existential variables $\mathbf{y} = (y_1, ..., y_m)$ is a formula of the form

$$\Phi = \forall x_1 ... \forall x_n \exists y_1(x_{d_{1,1}}, ..., x_{d_{1,n_1}}) ... \exists y_m(x_{d_{m,1}}, ..., x_{d_{m,n_m}})\ \phi(\mathbf{x}, \mathbf{y})$$

In Sections 4 and 5, we will also allow free variables, using the same notation and definition of equivalence as for QBF^*.

The class *DQHORN* contains all *DQBF* formulas in conjunctive normal form (*CNF*) whose clauses have at most one positive literal.

As stated in the following Definitions 1 and 2, the semantics of *DQBF* is defined over *model functions.* A *DQBF* formula is said to be true if for each existential variable y_i, there exists a propositional formula f_{y_i} over the universals $x_{d_{i,1}}, ..., x_{d_{i,n_i}}$ on which y_i depends, such that substituting the model functions for the existential variables (and dropping the existential quantifiers) leads to a universally quantified *QBF* formula which is true. The tuple $M = (f_{y_1}, ..., f_{y_m})$ of such functions is called a *satisfiability model.*

Definition 1. *For a dependency quantified Boolean formula $\Phi \in DQBF$ with existential variables $\mathbf{y} = (y_1, ..., y_m)$, let $M = (f_{y_1}, ..., f_{y_m})$ be a mapping which maps each existential variable y_i to a propositional formula f_{y_i} over the universal variables $x_{d_{i,1}}, ..., x_{d_{i,n_i}}$ on which y_i depends. Then M is a* ***satisfiability model*** *for Φ if the resulting QBF formula $\Phi[\mathbf{y}/M] := \Phi[y_1/f_{y_1}, ..., y_m/f_{y_m}]$, where si-*

multaneously each existential variable y_i is replaced by its corresponding formula f_{y_i} and the existential quantifiers are dropped from the prefix, is true.

Definition 2. *A dependency quantified Boolean formula is true if and only if it has a satisfiability model.*

The notion of satisfiability models has been originally introduced in [12] for *QBF* formulas. For *QBF*s, the last definition is actually a theorem, because their semantics is usually defined inductively without referring to model functions, which is not possible for *DQBF*s. In fact, the NEXPTIME-completeness of *DQBF* suggests that solving a *DQBF* formula involves finding and storing those functions. Fortunately, we will soon see that this is not a problem in the *DQHORN* case.

3 Satisfiability Models for *DQHORN* Formulas

We are not only interested in the mere existence of satisfiability models, but we also want to characterize their structure for certain classes of formulas. In this section, we will see that *DQHORN* formulas have satisfiability models of a very simple structure.

We begin with an observation: it is a well known fact about *propositional* Horn formulas, proved by Alfred Horn himself [10], that the intersection of two satisfying truth assignments is a satisfying truth assignment, too. If we represent truth assignments by sets which collect the variables that are assigned the value 1, the intersection of these assignments is given by the intersection of the corresponding sets of variables.

Now assume that a quantified Horn formula with two universal variables x_i and x_j is known to be satisfiable when $x_i = 0$ and $x_j = 1$ or when $x_i = 1$ and $x_j = 0$. That means there exist two truth assignments t_1 and t_2 to the existential variables such that the formula is satisfied in both cases. If we could lift the closure under intersection to the quantified case, it would mean that the intersection of t_1 and t_2 would satisfy the formula when both x_i and x_j are zero. This would imply that the satisfiability of a quantified Horn formula is determined only by those cases where at most one of the universal variables is zero.

Unfortunately, we have to obey the quantifier dependencies when choosing truth values for the existential variables, so we cannot simply intersect t_1 and t_2. Thus, lifting this result is obviously not so straightforward for the *QHORN* case, and even less straightforward for *DQHORN* with its sophisticated dependencies. What we need here is a way to characterize the behavior of the existentially quantified variables. As it turns out, satisfiability models are a suitable formalism for this and allow us to present a model-based proof which even works for *DQHORN*.

Since the number of zeros being assigned to the universal variables is an important criterion for our investigations, we first introduce some useful notation.

Definition 3. *By B_n^i, we denote the bit vector of length n where only the i-th element is zero, i.e. $B_n^i := (b_1, ..., b_n)$ with $b_i = 0$ and $b_j = 1$ for $j \neq i$.*
Moreover, we define the following relations on n-tuples of truth values:

1. $Z_{\leq 1}(n) = \bigcup_i \{B_n^i\} \cup \{(1, ..., 1)\}$ *(at most one zero)*
2. $Z_{=1}(n) = \bigcup_i \{B_n^i\}$ *(exactly one zero)*
3. $Z_{\geq 1}(n) = \{(a_1, ..., a_n) \mid \exists i : a_i = 0\}$ *(at least one zero)*

For example, if $n = 3$, we have the following relations:

$$Z_{\leq 1}(3) = \{(0,1,1), (1,0,1), (1,1,0), (1,1,1)\}$$
$$Z_{=1}(3) = \{(0,1,1), (1,0,1), (1,1,0)\}$$
$$Z_{\geq 1}(3) = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0)\}$$

We omit the parameter n and simply write $Z_{\leq 1}$ (or $Z_{=1}$ resp. $Z_{\geq 1}$) when it is clear from the context. Usually, n equals the total number of the universal quantifiers in a given formula.

Let $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in DQBF$. The definition of a satisfiability model in Section 2 requires that substituting the existentials $\mathbf{y}$ in Φ produces a formula $\Phi[\mathbf{y}/M]$ which is true. That means the matrix $\phi[\mathbf{y}/M]$ must be true for all possible assignments to the universals $\mathbf{x}$. But as motivated before, we want to focus only on the cases where at most one of the universals is assigned zero. Accordingly, we now introduce a special kind of satisfiability model which weakens the condition that all possible assignments are considered: a so-called $R_\forall$*-partial satisfiability model* is only required to satisfy $\phi[\mathbf{y}/M]$ for certain truth assignments to the universal variables which are given by a relation $R_\forall$.

Definition 4. *For a formula $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in DQBF$ with universal variables $\mathbf{x} = (x_1, ..., x_n)$ and existential variables $\mathbf{y} = (y_1, ..., y_m)$, let $M = (f_{y_1}, ..., f_{y_m})$ be a mapping which maps each existential variable y_i to a propositional formula f_{y_i} over the universal variables $x_{d_{i,1}}, ..., x_{d_{i,n_i}}$ on which y_i depends. Furthermore, let $R_\forall(n)$ be a relation on the set of possible truth assignments to n universals. Then M is a* ***$R_\forall$-partial satisfiability model*** *for Φ if the formula $\phi[\mathbf{y}/M]$ is true for all $\mathbf{x} \in R_\forall(n)$.*

Consider the following example:

$$\Phi = \forall x_1 \forall x_2 \forall x_3 \exists y_1(x_1, x_2) \exists y_2(x_2, x_3)\,(x_1 \vee y_1) \wedge (x_2 \vee \neg y_1) \wedge (\neg x_2 \vee x_3 \vee \neg y_2)$$

Then Φ does not have a satisfiability model, but $M = (f_{y_1}, f_{y_2})$ with $f_{y_1}(x_1, x_2) = \neg x_1 \vee x_2$ and $f_{y_2}(x_2, x_3) = 0$ is a $Z_{\leq 1}$-partial satisfiability model for Φ, because $\phi[\mathbf{y}/M] = (x_1 \vee \neg x_1 \vee x_2) \wedge (x_2 \vee (x_1 \wedge \neg x_2)) \wedge (\neg x_2 \vee x_3 \vee 1) \approx x_2 \vee x_1$, which is true for all $\mathbf{x} = (x_1, x_2, x_3)$ with $\mathbf{x} \in Z_{\leq 1}$.

It is not surprising that the mere existence of a $Z_{\leq 1}$-partial satisfiability model does not imply the existence of a (total) satisfiability model - at least not in the general case. But as discussed before, we are going to prove that for *DQHORN* formulas, the behavior of the formula for $\mathbf{x} \in Z_{\leq 1}$ does indeed completely determine its satisfiability. Accordingly, we now show: if we can find a $Z_{\leq 1}$-partial satisfiability model M to satisfy a *DQHORN* formula whenever at most one of the universals is false, then we can also satisfy the formula for arbitrary truth assignments to the universals. We achieve this by using M to construct a (total) satisfiability model M^t.

Definition 5. *Let $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in DQHORN$ with universal variables $\mathbf{x} = (x_1, ..., x_n)$ and existential variables $\mathbf{y} = (y_1, ..., y_m)$, and let $M = (f_{y_1}, .., f_{y_m})$ be a $Z_{\leq 1}$-partial satisfiability model for Φ. For each $f_{y_i}(x_{d_{i,1}}, ..., x_{d_{i,n_i}})$ in M, we define $f^t_{y_i}$ as follows:*

$$\begin{aligned} f^t_{y_i}(x_{d_{i,1}}, ..., x_{d_{i,n_i}}) := \quad & (\neg x_{d_{i,1}} \rightarrow f_{y_i}(0, 1, 1, ..., 1)) \\ & \wedge (\neg x_{d_{i,2}} \rightarrow f_{y_i}(1, 0, 1, ..., 1)) \\ & \wedge ... \\ & \wedge (\neg x_{d_{i,n_i}} \rightarrow f_{y_i}(1, 1, ..., 1, 0)) \\ & \wedge f_{y_i}(1, ..., 1) \end{aligned}$$

*Then we call $M^t = (f^t_{y_1}, ..., f^t_{y_m})$ the **total completion** of M.*

The intuition behind this definition is the following: for *each* argument which is zero, we consider the value of the original function when *only* this argument is zero. Then we return the conjunction (the intersection) of those original function values. Additionally, we have to intersect with $f_{y_i}(1, ..., 1)$. For example, $f^t_{y_i}(1, 0, 0, 1) = f_{y_i}(1, 0, 1, 1) \wedge f_{y_i}(1, 1, 0, 1) \wedge f_{y_i}(1, 1, 1, 1)$. In case all the arguments are 1, we simply return the value of the original function, i.e. $f^t_{y_i}(1, .., 1) = f_{y_i}(1, .., 1)$.

At the beginning of this section, we have mentioned that for propositional Horn formulas, the intersection of satisfying truth assignments is again a satisfying truth assignment. If you compare this to the previous definition (together with the following theorem), you will notice that we have just presented the generalized *DQHORN* version of it. The most important difference is that we now always intersect with $f_{y_i}(1, ..., 1)$. This takes care of the cases where certain universal variables are zero, but y_i does not depend on them due to the imposed quantifier dependencies.

Theorem 1. *Let $\Phi = Q\phi(\mathbf{x}, \mathbf{y}) \in DQHORN$ be a dependency quantified Horn formula with a $Z_{\leq 1}$-partial satisfiability model $M = (f_{y_1}, .., f_{y_m})$. Then the total completion of M, i.e. $M^t = (f^t_{y_1}, ..., f^t_{y_m})$ as defined above, is a satisfiability model for Φ.*

Proof. We must show that $\phi[\mathbf{y}/M^t]$ is true for all truth assignments $t(\mathbf{x}) := (t(x_1), ..., t(x_n)) \in \{0, 1\}^n$ to the universal variables.
Since $f^t_{y_j}(1, ..., 1) = f_{y_j}(1, ..., 1)$, we only need to consider $t(\mathbf{x}) \in Z_{\geq 1}$.

The proof is by induction on the number of zeros in $t(\mathbf{x})$. The induction base is the case $t(\mathbf{x}) \in Z_{=1}$. Then the definition of M^t implies that

$$f^t_{y_j}(t(x_{d_{j,1}}), ..., t(x_{d_{j,n_j}})) = f_{y_j}(t(x_{d_{j,1}}), ..., t(x_{d_{j,n_j}})) \wedge f_{y_j}(1, ..., 1) = 1$$

for all y_j. Now let $t(\mathbf{x}) = B_n^i$ be an assignment to the universals where $t(x_i) = 0$. In order to prove that every clause in $\phi[\mathbf{y}/M^t]$ is true for $t(\mathbf{x})$, we make a case distinction on the structure of Horn clauses. Any clause C belongs to one of the following cases:

1. C contains a positive existential variable y_j:
   Consider a clause of the form $C = y_j \vee \bigvee_{l \in L_y} \neg y_l \vee \bigvee_{l \in L_x} \neg x_l$. We assume that $i \notin L_x$, because $C[\mathbf{y}/M^t]$ is trivially true for $t(\mathbf{x})$ if $i \in L_x$.
   If $f_{y_j}(t(x_{d_{j,1}}), ..., t(x_{d_{j,n_j}})) = f_{y_j}(1, ..., 1) = 1$ then $f_{y_j}^t(t(x_{d_{j,1}}), ..., t(x_{d_{j,n_j}})) = 1$. Otherwise, without loss of generality, let $f_{y_j}(t(x_{d_{j,1}}), ..., t(x_{d_{j,n_j}})) = 0$. Then $f_{y_r}(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$ for some $r \in L_y$, as M is a $Z_{\leq 1}$-partial satisfiability model. This implies $f_{y_r}^t(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$, which makes $C[\mathbf{y}/M^t]$ true.
2. C contains a positive universal variable x_j:
   Consider a clause of the form $C = x_j \vee \bigvee_{l \in L_x} \neg x_l \vee \bigvee_{l \in L_y} \neg y_l$. The only interesting case to discuss is $i = j$. As above, M being a $Z_{\leq 1}$-partial satisfiability model implies that $f_{y_r}(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$ for some $r \in L_y$. And this implies $f_{y_r}^t(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$.
3. no positive literal in C:
   Consider a clause of the form $C = \bigvee_{l \in L_x} \neg x_l \vee \bigvee_{l \in L_y} \neg y_l$. We only need to discuss the case that $i \notin L_x$. Again, M being a $Z_{\leq 1}$-partial satisfiability model implies that we have $f_{y_r}(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$ for some $r \in L_y$. This means $f_{y_r}^t(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$.

For the induction step, we consider an assignment where $k > 1$ universals are false. Let $t(x_{i_1}) = 0$, ..., $t(x_{i_k}) = 0$ and $t(x_s) = 1$ for $s \neq i_1, ..., i_k$. In order to show that $\phi[\mathbf{y}/M^t]$ is true for $t(\mathbf{x})$, we can use the induction hypothesis and assume that $\phi[\mathbf{y}/M^t]$ is true for $t_1(\mathbf{x}) = B_n^{i_k}$ as well as for $t_{k-1}(\mathbf{x})$ with $t_{k-1}(x_1) = 0$, ..., $t_{k-1}(x_{i_{k-1}}) = 0$ and $t_{k-1}(x_s) = 1$ for $s \neq i_1, ..., i_{k-1}$. That means, the case with k zeros x_{i_1},..., x_{i_k} is reduced to the case where only x_{i_k} is zero and the case where x_{i_1}, ..., $x_{i_{k-1}}$ are zero. Then the definition of $f_{y_j}^t$ implies

$$f_{y_j}^t(t(x_{d_{j,1}}), ..., t(x_{d_{j,n_j}})) = f_{y_j}^t(t_1(x_{d_{j,1}}), ..., t_1(x_{d_{j,n_j}})) \wedge f_{y_j}^t(t_{k-1}(x_{d_{j,1}}), ..., t_{k-1}(x_{d_{j,n_j}}))$$

Again, we make a case distinction. It is actually very similar to the one from the induction base:

1. C contains a positive existential variable y_j:
   Consider a clause of the form $C = y_j \vee \bigvee_{l \in L_y} \neg y_l \vee \bigvee_{l \in L_x} \neg x_l$. We assume that $i_1, ..., i_k \notin L_x$, because otherwise, $C[\mathbf{y}/M^t]$ is trivially true for $t(\mathbf{x})$.
   If $f_{y_j}^t(t_1(x_{d_{j,1}}), ..., t_1(x_{d_{j,n_j}})) = 1$ and $f_{y_j}^t(t_{k-1}(x_{d_{j,1}}), ..., t_{k-1}(x_{d_{j,n_j}})) = 1$, we have $f_{y_j}^t(t(x_{d_{j,1}}), ..., t(x_{d_{j,n_j}})) = 1$.
   Otherwise, without loss of generality, let $f_{y_j}^t(t_1(x_{d_{j,1}}), ..., t_1(x_{d_{j,n_j}})) = 0$. Then the induction hypothesis implies that $f_{y_r}^t(t_1(x_{d_{r,1}}), ..., t_1(x_{d_{r,n_r}})) = 0$ for some $r \in L_y$, and we get $f_{y_r}^t(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$.

2. C contains a positive universal variable x_j:
   Consider a clause of the form $C = x_j \vee \bigvee_{l \in L_x} \neg x_l \vee \bigvee_{l \in L_y} \neg y_l$. The only interesting case to discuss is $j \in \{i_1, ..., i_k\}$. Without loss of generality, we assume $j = i_k$.
   It follows from the induction hypothesis that $f^t_{y_r}(t_1(x_{d_{r,1}}), ..., t_1(x_{d_{r,n_r}})) = 0$ for some $r \in L_y$. Then $f^t_{y_r}(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$.
3. no positive literal in C:
   Consider a clause of the form $C = \bigvee_{l \in L_x} \neg x_l \vee \bigvee_{l \in L_y} \neg y_l$. We only need to discuss the case that $i_1, ..., i_k \notin L_x$. Again, the induction hypothesis implies that we have $f^t_{y_r}(t_1(x_{d_{r,1}}), ..., t_1(x_{d_{r,n_r}})) = 0$ for some $r \in L_y$. Then $f^t_{y_r}(t(x_{d_{r,1}}), ..., t(x_{d_{r,n_r}})) = 0$.

□

4 From *DQBF** to *QBF**: Eliminating Universals

4.1 The General Case

Quantifier expansion is a well-known technique for solving *QBF*s [1, 3]. As demonstrated in this section, it can be generalized to dependency quantified formulas and may be used to compute for any *DQBF** formula an equivalent prenex *QBF** formula.

A universal quantifier $\forall x\ \phi(x)$ is just an abbreviation for $\phi(0) \wedge \phi(1)$, so we can expand it and make two copies of the original matrix, one for the universally quantified variable being false, and one for that variable being true. As explained in [3], existential variables which depend on that universal variable need to be duplicated as well. For example, in

$$\forall x_1 \forall x_2 \forall x_3 \exists y_1(x_1, x_2) \exists y_2(x_2, x_3)\, \phi(x_1, x_2, x_3, y_1, y_2)$$

the choice for y_1 depends on the value of x_1. We must therefore introduce two separate instances $y_{1,(0)}$ and $y_{1,(1)}$ of the original variable y_1, where $y_{1,(0)}$ is used in the copy of the matrix for $x_1 = 0$, and analogously $y_{1,(1)}$ for $x_1 = 1$. We obtain the expanded formula

$$\forall x_2 \forall x_3 \exists y_{1,(0)}(x_2) \exists y_{1,(1)}(x_2) \exists y_2(x_2, x_3)\, \phi(0, x_2, x_3, y_{1,(0)}, y_2) \wedge \phi(1, x_2, x_3, y_{1,(1)}, y_2)$$

We can do this successively to expand multiple universal quantifiers. Unlike the *QBF** case described in [3] and [4], we do not need to start with the innermost quantifier, because *DQBF** formulas can always be written with a $\forall^*\exists^*$ prefix where the order of the universals is irrelevant. After expanding all universal quantifiers, we are left with a *QBF** formula - actually a very special one with a $\exists^*$ prefix. Obviously, this expansion leads to an exponential blowup of the original formula. In practice, we do not need to expand all universals. For our sample formula, the expansion of x_1 is sufficient, because the resulting formula can be written in *QBF** as

$$\forall x_2 \exists y_{1,(0)} \exists y_{1,(1)} \forall x_3 \exists y_2\, \phi(0, x_2, x_3, y_{1,(0)}, y_2) \wedge \phi(1, x_2, x_3, y_{1,(1)}, y_2)$$

One could think of sophisticated strategies for selecting which universals must be expanded for a given *DQBF** formula. In the general case, however, this cannot avoid exponential growth, therefore the following discussion will assume that all universal quantifiers are eliminated. Using the results from the previous section, we will show that this is not a problem for *DQHORN** formulas, because the expanded formula is always small, even if all universals are expanded.

In the general case, for a $DQBF^*$ formula

$$\Phi(\mathbf{z}) = \forall x_1 ... \forall x_n \exists y_1(x_{d_{1,1}}, ..., x_{d_{1,n_1}}) ... \exists y_m(x_{d_{m,1}}, ..., x_{d_{m,n_m}})\, \phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

with universal variables $\mathbf{x} = x_1, ..., x_n$, existential variables $\mathbf{y} = y_1, ..., y_m$ and free variables $\mathbf{z}$, we obtain the expanded *QBF** formula

$$\begin{aligned} \Phi_{\exists QBF}(\mathbf{z}) \quad := \quad & \exists y_{1,(0,...,0)} \exists y_{1,(0,...,0,1)} ... \exists y_{1,(1,...,1,0)} \exists y_{1,(1,...,1)} \\ & \qquad \vdots \\ & \exists y_{m,(0,...,0)} \exists y_{m,(0,...,0,1)} ... \exists y_{m,(1,...,1,0)} \exists y_{m,(1,...,1)} \\ \bigwedge_{t(\mathbf{x}) \in \{0,1\}^n} & \phi(t(\mathbf{x}), y_{1,(t(x_{d_{1,1}}),...,t(x_{d_{1,n_1}}))}, ..., y_{m,(t(x_{d_{m,1}}),...,t(x_{d_{m,n_m}}))}, \mathbf{z}) \end{aligned}$$

We omit the formal proof that $\Phi(\mathbf{z}) \approx \Phi_{\exists QBF}(\mathbf{z})$, as it is quite obvious that $\Phi_{\exists QBF}$ is simply the formalization of the elimination algorithm we have just described.

Here is an example: the formula

$$\Phi(\mathbf{z}) = \forall x_1 \forall x_2 \forall x_3 \exists y_1(x_1, x_2) \exists y_2(x_2, x_3)\, \phi(x_1, x_2, x_3, y_1, y_2, \mathbf{z})$$

from above is expanded to

$$\begin{aligned} \Phi_{\exists QBF}(\mathbf{z}) \;=\; & \exists y_{1,(0,0)} \exists y_{1,(0,1)} \exists y_{1,(1,0)} \exists y_{1,(1,1)} \exists y_{2,(0,0)} \exists y_{2,(0,1)} \exists y_{2,(1,0)} \exists y_{2,(1,1)} \\ & \phi(0,0,0, y_{1,(0,0)}, y_{2,(0,0)}, \mathbf{z}) \wedge \phi(0,0,1, y_{1,(0,0)}, y_{2,(0,1)}, \mathbf{z}) \\ \wedge\; & \phi(0,1,0, y_{1,(0,1)}, y_{2,(1,0)}, \mathbf{z}) \wedge \phi(0,1,1, y_{1,(0,1)}, y_{2,(1,1)}, \mathbf{z}) \\ \wedge\; & \phi(1,0,0, y_{1,(1,0)}, y_{2,(0,0)}, \mathbf{z}) \wedge \phi(1,0,1, y_{1,(1,0)}, y_{2,(0,1)}, \mathbf{z}) \\ \wedge\; & \phi(1,1,0, y_{1,(1,1)}, y_{2,(1,0)}, \mathbf{z}) \wedge \phi(1,1,1, y_{1,(1,1)}, y_{2,(1,1)}, \mathbf{z}) \end{aligned}$$

4.2 Special Case: *DQHORN**

We will now show that the expansion of universal quantifiers is feasible for $DQHORN^*$ formulas.

Definition 6. *Let $\Phi \in DQHORN^*$ with*

$$\Phi(\mathbf{z}) = \forall x_1 ... \forall x_n \exists y_1(x_{d_{1,1}}, ..., x_{d_{1,n_1}}) ... \exists y_m(x_{d_{m,1}}, ..., x_{d_{m,n_m}})\, \phi(\mathbf{x}, \mathbf{y}, \mathbf{z})$$

be a dependency quantified Horn formula with universal variables $\mathbf{x} = x_1, ..., x_n$, existential variables $\mathbf{y} = y_1, ..., y_m$ and free variables $\mathbf{z}$.

Then we define the formula $\Phi_{\exists HORN}(\mathbf{z})$ as

$$\begin{aligned}\Phi_{\exists HORN}(\mathbf{z}) \;:=\; & \exists y_{1,(0,1,\ldots,1)} \exists y_{1,(1,0,1,\ldots,1)} \ldots \exists y_{1,(1,\ldots,1,0)} \exists y_{1,(1,\ldots,1)} \\ & \vdots \\ & \exists y_{m,(0,1,\ldots,1)} \exists y_{m,(1,0,1,\ldots,1)} \ldots \exists y_{m,(1,\ldots,1,0)} \exists y_{m,(1,\ldots,1)} \\ \bigwedge_{t(\mathbf{x}) \in Z_{\leq 1}(n)} & \phi(t(\mathbf{x}), y_{1,(t(x_{d_{1,1}}),\ldots,t(x_{d_{1,n_1}}))}, \ldots, y_{m,(t(x_{d_{m,1}}),\ldots,t(x_{d_{m,n_m}}))}, \mathbf{z})\end{aligned}$$

The only difference between the formula $\Phi_{\exists HORN}$ and the expansion $\Phi_{\exists QBF}$ for general *DQBF** formulas is that for Horn formulas, not all possible truth assignments to the universally quantified variables have to be considered. Based on the results from Section 3, assignments where more than one universal variable is false are irrelevant for *DQHORN** formulas.

For the formula

$$\Phi(\mathbf{z}) = \forall x_1 \forall x_2 \forall x_3 \exists y_1(x_1, x_2) \exists y_2(x_2, x_3)\, \phi(x_1, x_2, x_3, y_1, y_2, \mathbf{z})$$

from the example in Section 4.1, we have

$$\begin{aligned}\Phi_{\exists HORN}(\mathbf{z}) \;=\; & \exists y_{1,(0,1)} \exists y_{1,(1,0)} \exists y_{1,(1,1)} \exists y_{2,(0,1)} \exists y_{2,(1,0)} \exists y_{2,(1,1)} \\ & \phi(0,1,1,y_{1,(0,1)}, y_{2,(1,1)}, \mathbf{z}) \wedge \phi(1,0,1,y_{1,(1,0)}, y_{2,(0,1)}, \mathbf{z}) \\ \wedge\; & \phi(1,1,0,y_{1,(1,1)}, y_{2,(1,0)}, \mathbf{z}) \wedge \phi(1,1,1,y_{1,(1,1)}, y_{2,(1,1)}, \mathbf{z})\end{aligned}$$

Before we can prove that $\Phi_{\exists HORN}$ is indeed equivalent to Φ, we make a fundamental observation: for the special case that Φ is closed, i.e. there are no free variables, the satisfiability of $\Phi_{\exists HORN}$ implies the existence of a $Z_{\leq 1}$-partial satisfiability model for Φ.

Lemma 1. *Let $\Phi \in DQHORN$ be a dependency quantified Horn formula without free variables, and let $\Phi_{\exists HORN}$ be defined as above. If $\Phi_{\exists HORN}$ is satisfiable then Φ has a $Z_{\leq 1}$-partial satisfiability model.*

Proof. Let t be a satisfying truth assignment to the existentials in $\Phi_{\exists HORN}$. This assignment t provides us with all the information needed to construct a $Z_{\leq 1}$-partial satisfiability model for Φ. The idea is to assemble the truth assignments to the individual copies $y_{i,(x_{d_{i,1}},\ldots,x_{d_{i,n_i}})}$ of an existential variable y_i into a common model function. We achieve this with the following definition:

$$\begin{aligned}f_{y_i}(x_{d_{i,1}}, \ldots, x_{d_{i,n_i}}) = \;& (\bar{x}_{d_{i,1}} \wedge x_{d_{i,2}} \wedge \ldots \wedge x_{d_{i,n_i}} \;\rightarrow t(y_{i,(0,1,\ldots,1)})) \\ & \wedge (x_{d_{i,1}} \wedge \bar{x}_{d_{i,2}} \wedge x_{d_{i,3}} \wedge \ldots \wedge x_{d_{i,n_i}} \;\rightarrow t(y_{i,(1,0,1,\ldots,1)})) \\ & \wedge \ldots \\ & \wedge (x_{d_{i,1}} \wedge \ldots \wedge x_{d_{i,n_i-1}} \wedge \bar{x}_{d_{i,n_i}} \;\rightarrow t(y_{i,(1,\ldots,1,0)})) \\ & \wedge (x_{d_{i,1}} \wedge \ldots \wedge x_{d_{i,n_i}} \;\rightarrow t(y_{i,(1,\ldots,1)}))\end{aligned}$$

Now, the f_{y_i} form a $Z_{\leq 1}$-partial satisfiability model for Φ, because for all $\mathbf{x} = (x_1, \ldots, x_n)$ with $\mathbf{x} \in Z_{\leq 1}$, we have $f_{y_i}(x_{d_{i,1}}, \ldots, x_{d_{i,n_i}}) = t(y_{i,(x_{d_{i,1}},\ldots,x_{d_{i,n_i}})})$, and $\phi(x_1, \ldots, x_n, t(y_{1,(x_{d_{1,1}},\ldots,x_{d_{1,n_1}})}), \ldots, t(y_{m,(x_{d_{m,1}},\ldots,x_{d_{m,n_m}})})) = 1$ due to the satisfiability of $\Phi_{\exists HORN}$. □

Using Lemma 1 in combination with Theorem 1, it is now easy to show that $\Phi_{\exists HORN}$ is equivalent to Φ.

Theorem 2. *For $\Phi \in DQHORN^*$ and $\Phi_{\exists HORN}$ as defined above, it holds that $\Phi \approx \Phi_{\exists HORN}$.*

Proof. The implication $\Phi(\mathbf{z}) \models \Phi_{\exists HORN}(\mathbf{z})$ is obvious, as the clauses in $\Phi_{\exists HORN}$ are just a subset of the clauses in $\Phi_{\exists QBF}$, which in turn is equivalent to Φ.
The implication $\Phi_{\exists HORN}(\mathbf{z}) \models \Phi(\mathbf{z})$ is more interesting. Assume $\Phi_{\exists HORN}(\mathbf{z}^*)$ is satisfiable for some fixed $\mathbf{z}^*$. With the free variables fixed, we can treat both $\Phi_{\exists HORN}(\mathbf{z}^*)$ and $\Phi(\mathbf{z}^*)$ as closed formulas and apply Lemma 1 and the results from Section 3 as follows:
According to Lemma 1, the satisfiability of $\Phi_{\exists HORN}(\mathbf{z}^*)$ implies that $\Phi(\mathbf{z}^*)$ has a $Z_{\leq 1}$-partial satisfiability model. On this partial model, we can apply the total expansion from Definition 5 and Theorem 1 to obtain a (total) satisfiability model. The fact that $\Phi(\mathbf{z}^*)$ has a satisfiability model implies that $\Phi(\mathbf{z}^*)$ is satisfiable. □

We immediately obtain the following corollary:

Corollary 1. *For any dependency quantified Horn formula $\Phi \in DQHORN^*$ with free variables, there exists an equivalent formula $\Phi_{\exists HORN} \in QHORN^*$ without universal quantifiers. The length of $\Phi_{\exists HORN}$ is bounded by $|\forall| \cdot |\Phi|$, where $|\forall|$ is the number of universal quantifiers in Φ, and $|\Phi|$ is the length of Φ.*

5 Solving *DQHORN**-SAT

We can take advantage of the fact that the transformation we have just presented produces $QHORN^*$ formulas without universal variables. The absence of universals allows us to easily determine their satisfiability, because a formula of the form $\Psi(\mathbf{z}) = \exists y_1 ... \exists y_m \, \psi(y_1, ..., y_m, \mathbf{z})$ is satisfiable if and only if its matrix $\psi(y_1, ..., y_m, \mathbf{z})$ is satisfiable. The latter is a purely propositional formula, so we can apply existing SAT solvers for propositional Horn.

We then obtain the following algorithm for determining the satisfiability of a formula $\Phi \in DQHORN^*$:

1. Transform Φ into $\Phi_{\exists HORN}$ according to Definition 6. This requires time $O(|\forall| \cdot |\Phi|)$ and produces a formula of length $|\Phi_{\exists HORN}| = O(|\forall| \cdot |\Phi|)$.
2. Determine the satisfiability of $\phi_{\exists HORN}$, which is the purely propositional matrix of $\Phi_{\exists HORN}$. It is well known [6] that SAT for propositional Horn formulas can be solved in linear time, in this case $O(|\phi_{\exists HORN}|) = O(|\forall| \cdot |\Phi|)$.

In total, this requires time $O(|\forall| \cdot |\Phi|)$, which is just as hard as $QHORN^*$-SAT [11].

6 Conclusion

We have introduced the class of dependency quantified Horn formulas $DQHORN^*$ and have shown that it is a tractable subclass of $DQBF^*$. We have demonstrated that the tractability of $DQHORN^*$ is due to an interesting effect that the Horn property has on the behavior of the quantifiers, a phenomenon which is preserved when adding partially ordered quantifiers. Based on this result, we have been able to prove that

- any dependency quantified Horn formula $\Phi \in DQHORN^*$ of length $|\Phi|$ with free variables, $|\forall|$ universal quantifiers and an arbitrary number of existential quantifiers can be transformed into an equivalent quantified Horn formula of length $O(|\forall| \cdot |\Phi|)$ which contains only existential quantifiers.
- $DQHORN^*$-SAT can be solved in time $O(|\forall| \cdot |\Phi|)$.

This shows that the class $DQHORN^*$ is no more difficult than $QHORN^*$, but apparently does not provide significant increases in expressive power either. $DQHORN^*$should, however, not be considered as an isolated subclass of $DQBF^*$. Just like ordinary $QHORN^*$ formulas are important as subproblems when solving arbitrary QBF^* formulas [5], our findings on $DQHORN^*$ should prove useful for handling more general classes of $DQBF^*$ formulas. And since the latest trend of enabling QBF solvers to directly handle non-prenex formulas [8] constitutes a special case of partially-ordered quantification with tree-like dependencies, our results might also be applied in non-prenex QBF solvers for cutting Horn branches.

In addition, the tractability of $DQHORN^*$ shows that adding partially ordered quantifiers does not necessarily lead to a computational blow-up as in the general case with $DQBF^*$. Further research should therefore explore the complexity and expressiveness of other subclasses and special cases, in particular tree-like dependencies as mentioned above.

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