Rewriting (Dependency-)Quantified 2-CNF with Arbitrary Free Literals into Existential 2-HORN

Uwe Bubeck and Hans Kleine Büning *

Universität Paderborn, Germany. bubeck@upb.de, kbcsl@upb.de

Abstract. We extend quantified 2-CNF formulas by also allowing literals over free variables which are exempt from the 2-CNF restriction. That means we consider quantified CNF formulas with clauses that contain at most two bound literals and an arbitrary number of free literals. We show that these $Q2\text{-}CNF^b$ formulas can be transformed in polynomial time into purely existentially quantified CNF formulas in which the bound literals are in 2-HORN ($\exists 2\text{-}HORN^b$).
Our result still holds if we allow Henkin-style quantifiers with explicit dependencies. In general, dependency quantified Boolean formulas ($DQBF$) are assumed to be more succinct at the price of a higher complexity. This paper shows that $DQ2\text{-}CNF^b$ has a similar expressive power and complexity as $\exists 2\text{-}HORN^b$. In the special case that the 2-CNF restriction is also applied to the free variables ($DQ2\text{-}CNF^*$), the satisfiability can be decided in linear time.

1 Introduction

Quantified Boolean formulas (QBF) generalize propositional formulas by allowing variables to be quantified universally or existentially. In this paper, we also allow free variables which are not quantified and indicate this with a star (QBF^*). An interesting property of quantified Boolean formulas with free variables is that it is possible to define an equivalence between such formulas and propositional formulas. We say that $\Phi \in QBF^*$ is equivalent to $\psi \in PROP$ ($\Phi \approx \psi$) if and only if the free variables in Φ correspond to the propositional variables in ψ and both formulas have the same truth value for each assignment to the free/propositional variables. This means that quantified variables inside of Φ are not taken into consideration here, so these can be thought of as local or auxiliary variables. An important application of auxiliary variables is to introduce abbreviations for repeating parts in a given formula, such as multiple copies of transition or reachability relations in verification problems [9, 14]. Accordingly, QBF^* representations are often much more compact than equivalent propositional encodings, in addition to the advantage that many problems have a natural forall-exists semantics which can elegantly be modeled by quantifiers [20].

* Partially supported by the German Research Foundation (DFG), grant KL 529/QBF

Unfortunately, quantified Boolean formulas appear to be much harder to solve than propositional formulas, with *QBF* and *QBF** satisfiability being *PSPACE*-complete. This makes it worthwhile to investigate subclasses with a lower decision complexity. An interesting idea is to consider *QBF** formulas in clausal form with additional restrictions only on the quantified literals. Let $\Phi = Q \bigwedge_i (\phi_i^b \vee \phi_i^f)$ be a quantified Boolean formula with quantifiers Q, such that ϕ_i^b is a clause over bound variables (called *bound part*) and ϕ_i^f a clause over free variables (the *free part*). Then we require that $Q \bigwedge_i \phi_i^b \in QK$ for a formula class QK, while the free parts ϕ_i^f may have arbitrary structure. Such formulas, which we call QK^b for a base class QK, can be surprisingly powerful.

For example, $QHORN^b$ denotes quantified Horn formulas in which the Horn property is only enforced on the quantified variables, which means each clause has at most one positive and arbitrarily many negative literals over quantified variables, but an arbitrary number of free literals with arbitrary polarity. Obviously, every propositional *CNF* formula is also a $QHORN^b$ formula, but this class is significantly more capable. For example, $QHORN^b$ formulas can compactly encode Boolean circuits with arbitrary fan-out (and vice versa) [1, 15], while it is generally assumed that there exist circuits with fan-out greater than 1 for which every equivalent propositional formula is exponentially larger. Furthermore, while there are propositional formulas for which every equivalent *CNF* formula is exponential, every propositional formula has a poly-size equivalent $QHORN^b$ formula, e.g. by the one-sided Tseitin transformation [22, 19] when the newly introduced variables are bound by existential quantifiers. In fact, such poly-size *CNF* transformations can even be accomplished with $\exists$*2-HORN*^{b} formulas, that is, existentially quantified formulas in clausal form with at most two bound literals per clause, one of which may be positive [8]. At the same time, $QHORN^b$ satisfiability is not significantly more difficult than propositional satisfiability, because the universal quantifiers can easily be eliminated [15], which makes $QHORN^b$ satisfiability *NP*-complete.

Besides *HORN*, another standard restriction on the structure of clauses is *2-CNF*. The goal of this paper is to investigate the implications of enforcing a *2-CNF* restriction on the bound parts of *QBF** formulas in clausal form. That means we have clauses with at most two bound and arbitrarily many free literals, called *Q2-CNF*^{b} in line with the above terminology. This class is surprisingly powerful and indeed exponentially more expressive than propositional *CNF* because of the above remark about $\exists$*2-HORN*$^b \subseteq$ *Q2-CNF*^{b} formulas being sufficient for poly-size *CNF* transformation.

Normally, *2-CNF* formulas are not more difficult than *HORN* formulas. In the propositional case, it is well known that the satisfiability problem for both classes can be solved in linear time ([11, 2] and [13, 10]). For quantified *2-CNF* formulas with free variables, the satisfiability problem is still linear [2], whereas the best known algorithms for determining the satisfiability of a quantified Horn formula Φ with $|\forall|$ universal quantifiers require time $O(|\forall| \cdot |\Phi|)$ [12] ($|\Phi|$ is the length of Φ, counting all occurrences of variables, including those in quantifier definitions).

Is it possible to make similar statements about the complexity and expressive power of *Q2-CNF*^{b} in comparison to *QHORN*^{b} formulas? Our goal is to show that *Q2-CNF*^{b} formulas can be transformed in polynomial time into equivalent $\exists$*2-HORN*^{b} formulas. This immediately implies that *Q2-CNF*^{b} satisfiability is *NP*-complete, like *QHORN*^{b} satisfiability.

An intermediate result that we present is the elimination of all universal quantifiers from a *Q2-CNF*^{b} formula Φ in time and space $O(|\forall|^2|\Phi|)$. This might be useful for *QBF* solvers, since a *Q2-CNF*^{b} formula can be embedded as a subformula in a *QBF* formula if we consider variables which are bound by preceding quantifiers as free variables. For example, let $\Phi = Q((Q'\ \phi) \wedge \varphi) \approx QQ'\ (\phi \wedge \varphi)$ be a *QBF* formula in *CNF* where each clause in ϕ contains at most two literals over variables that are bound in Q', whereas the variables from Q can appear without restrictions in ϕ and φ. Then the transformation presented below allows the elimination of all universals in the *Q2-CNF*^{b} formula $Q'\ \phi$.

2 Dependency Quantified Boolean Formulas

In *QBF*, an existentially quantified variable can have different values depending on the values of universal variables whose quantifiers occur further outside. This imposes an ordering on the quantifiers where each existentially quantified variable depends on all preceding universal variables. Even if we waive the usual requirement that all quantifiers have to appear at the beginning in a dedicated quantifier prefix, it is not possible for two existential variables which occur in common clauses to depend on disjoint non-empty sets of universally quantified variables. Dependency quantified Boolean formulas (*DQBF* or *DQBF** with free variables) [18] make this possible by explicitly stating for each existentially quantified variable on which universals it depends. For example, $\Phi = \forall x_1 \forall x_2 \exists y_1(x_1) \exists y_2(x_2) \exists y_3(x_1, x_2)\ \phi(x_1, x_2, y_1, y_2, y_3)$ is a *DQBF* formula in which y_1 depends only on x_1, y_2 only on x_2 and y_3 on both x_1 and x_2.

Can we apply our poly-time transformation from *Q2-CNF*^{b} to $\exists$*2-CNF*^{b} also to *DQ2-CNF*^{b} formulas, which means formulas with dependency quantifiers as in the example above and at most two bound literals per clause? The fact that universal variables can be eliminated cheaply from *Q2-CNF*^{b} formulas implies that *2-CNF* is such a strong restriction that the ordering of quantifiers in the prefix loses much of its relevance. For *DQHORN*^{b}, the situation is similar: it is indeed possible to eliminate all dependency quantifiers with less than quadratic formula growth [6] (that proof is for *DQHORN**, but it also applies to *DQHORN*^{b}, since it does not rely on a particular structure of the free variables).

In general, however, *DQBF** encodings are assumed to be exponentially more compact in the best case than *QBF** encodings. Whereas *QBF* can be seen as a two-player game with an existential player reacting to moves of a universal player, *DQBF* corresponds to a three-player game where a universal player challenges two existential players with different inputs. Disjoint dependencies, like for y_1 and y_2 in the example above, guarantee that both existential players work independently. Such variables can still occur together in the same clauses, which is a vital feature that is not possible with *QBF*, even in non-prenex form. It allows the universal player to compare the results of independent existential players. This corresponds to a multi-prover interactive proof system [4], which is a very powerful concept, but also causes another jump in complexity, with *DQBF** satisfiability being *NEXPTIME*-complete [18].

Before we can develop a transformation from *DQ2-CNF*$^{\boldsymbol{b}}$ to $\exists$*2-CNF*$^{\boldsymbol{b}}$, we need a few basics. We require *DQBF** formulas to be in prenex form with a quantifier-free matrix, as negations of existential dependency quantifiers would be problematic. Because of the explicit dependencies, *DQBF** formulas can always be written with a $\forall^*\exists^*$ prefix. To quickly enumerate the dependencies of a given existential variable y_i, we use indices $d_{i,1},...,d_{i,n_i}$ which point to the n_i universals on which y_i depends. For example, given the existential quantifier $\exists y_4(x_3, x_5)$, we say that y_4 depends on $x_{d_{4,1}}$ and $x_{d_{4,2}}$ with $d_{4,1} = 3$ and $d_{4,2} = 5$. We also use a shorter notation $\exists y_i(\mathbf{x}_{\mathbf{d}_i})$ where we abbreviate $\mathbf{x}_{\mathbf{d}_i} := (x_{d_{i,1}},...,x_{d_{i,n_i}})$. It is allowed to have empty dependencies with $n_i = 0$, i.e. existential quantifiers $\exists y_i()$ that do not depend on any universals.

The semantics of *DQBF* and *DQBF** is defined by associating dependency quantified existentials $\exists y_i(x_{d_{i,1}},...,x_{d_{i,n_i}})$ with functions $f_{y_i}(x_{d_{i,1}},...,x_{d_{i,n_i}})$:

Definition 1. *(Satisfiability Model)*
For $\Phi \in DQBF$ with existential variables $\mathbf{y} = (y_1,...,y_m)$, let $M = (f_{y_1},...,f_{y_m})$ map each existential y_i to a propositional formula f_{y_i} over the universal variables $x_{d_{i,1}},...,x_{d_{i,n_i}}$ on which y_i depends.
*M is a **satisfiability model** for Φ if and only if $\Phi[\mathbf{y}/M] := \Phi[y_1/f_{y_1},...,y_m/f_{y_m}]$ is true, i.e. if a tautological formula is obtained when simultaneously each existential variable y_i is replaced with f_{y_i} and the existential quantifiers are dropped from the prefix.*

Definition 2. *(DQBF and DQBF* Semantics)*
*A DQBF formula Φ is **true** if and only if it has a satisfiability model.*
A DQBF formula $\Psi(\mathbf{z})$ with free variables $\mathbf{z} = (z_1,...,z_r)$ is **satisfiable** if and only if there exists a truth assignment $\tau(\mathbf{z}) = (\tau(z_1),...,\tau(z_r)) \in \{0,1\}^r$ to the free variables such that $\Psi(\tau(\mathbf{z})) \in DQBF$ is true, i.e. replacing all occurrences of free variables with their assigned truth value produces a true formula.*

3 Transformation from *DQ2-CNF*$^{\boldsymbol{b}}$ to $\exists$*2-CNF*$^{\boldsymbol{b}}$

There are two powerful concepts that we need for transforming *DQ2-CNF*$^{\boldsymbol{b}}$ formulas into $\exists$*2-CNF*$^{\boldsymbol{b}}$: universal expansion and minimal falsity/unsatisfiability.

Universal expansion in *QBF** is the elimination of universal quantifiers by the well-known equivalence $\forall x\ \Phi(x, \mathbf{z}) \approx \Phi(0, \mathbf{z}) \wedge \Phi(1, \mathbf{z})$, an operation which has been used successfully in various solvers, e.g. [3, 5, 7]. Care must be taken to duplicate also subsequent existential quantifiers which are in the scope of the expanded quantifier, in order to retain the ability to assign different values to an existential for different values of a preceding universal. In general, repeated application of this method obviously produces exponential formulas, even though the amount of duplication can often be significantly reduced in practice [5, 7, 17, 21]. We are going to show that the *2-CNF* restriction on the bound variables allows us to always apply universal expansion in a tractable fashion.

Universal expansion also works for *DQBF**, and the dependency lists immediately indicate which existentials must be duplicated when a universal variable is expanded. The correctness of universal expansion is bit more difficult to verify for *DQBF** because of the more implicit semantics definition by using model functions.

Lemma 1. *(Correctness of Universal Expansion for DQBF*)*
Let Φ be a DQBF formula in which we want to expand the universal quantifier $\forall x_n$. Without loss of generality, assume that the existentials are arranged in two blocks, depending on whether they are dominated by x_n or not:*

$$\begin{aligned}\Phi(\mathbf{z}) = \forall x_1 ... \forall x_n \exists y_1(\mathbf{x_{d_1}}) ... \exists y_k(\mathbf{x_{d_k}}) \exists y_{k+1}(\mathbf{x_{d_{k+1}}}, x_n) ... \exists y_m(\mathbf{x_{d_m}}, x_n)\\ \phi(x_1, ..., x_n, y_1, ..., y_m, \mathbf{z})\end{aligned}$$

with $x_n \notin \mathbf{x_{d_i}}$ for all $1 \leq i \leq m$. Then $\Phi(\mathbf{z}) \approx \Phi'(\mathbf{z})$ for the expanded formula

$$\begin{aligned}\Phi'(\mathbf{z}) = \forall x_1 ... \forall x_{n-1} \exists y_1(\mathbf{x_{d_1}}) ... \exists y_k(\mathbf{x_{d_k}})\\ \exists y_{k+1,(0)}, y_{k+1,(1)}(\mathbf{x_{d_{k+1}}}) ... \exists y_{m,(0)}, y_{m,(1)}(\mathbf{x_{d_m}})\\ \phi(x_1, ..., x_{n-1}, 0, y_1, ..., y_k, y_{k+1,(0)}, ..., y_{m,(0)}, \mathbf{z}) \wedge\\ \phi(x_1, ..., x_{n-1}, 1, y_1, ..., y_k, y_{k+1,(1)}, ..., y_{m,(1)}, \mathbf{z}) \ .\end{aligned}$$

Proof. We must prove that $\Phi(\tau(\mathbf{z})) = 1 \Leftrightarrow \Phi'(\tau(\mathbf{z})) = 1$ for any truth assignment $\tau(\mathbf{z}) := (\tau(z_1), ..., \tau(z_r)) \in \{0,1\}^r$ to the free variables $\mathbf{z} = (z_1, ..., z_r)$. For fixed $\tau(\mathbf{z})$, we can consider $\Phi(\tau(\mathbf{z}))$ and $\Phi'(\tau(\mathbf{z}))$ as closed *DQBF* formulas.

From left to right: let $M = (f_{y_1}, ..., f_{y_m})$ be a satisfiability model for $\Phi(\tau(\mathbf{z}))$. Define $G_{(0)} := (g_{y_1}, ..., g_{y_k}, g_{y_{k+1,(0)}}, ..., g_{y_{m,(0)}})$ with $g_{y_i} := f_{y_i}$ for $i = 1, ..., k$ and $g_{y_{i,(0)}}(x_{d_{i,1}}, ..., x_{d_{i,n_i}}) := f_{y_i}(x_{d_{i,1}}, ..., x_{d_{i,n_i}}, 0)$ for $i = k+1, ..., m$.
Then $\forall x_1 ... \forall x_{n-1}\ \phi(x_1, ..., x_{n-1}, 0, g_{y_1}, ..., g_{y_{m,(0)}}, \tau(\mathbf{z})) = 1$. With an analogous definition of $G_{(1)}$ with functions $g_{y_{i,(1)}}(x_{d_{i,1}}, ..., x_{d_{i,n_i}}) := f_{y_i}(x_{d_{i,1}}, ..., x_{d_{i,n_i}}, 1)$ for $i = k+1, ..., m$, $G = (g_{y_1}, ..., g_{y_k}, g_{y_{k+1,(0)}}, g_{y_{k+1,(1)}}, ..., g_{y_{m,(0)}}, g_{y_{m,(1)}})$ is a satisfiability model for $\Phi'(\tau(\mathbf{z}))$.

From right to left: let $G = (g_{y_1}, ..., g_{y_k}, g_{y_{k+1,(0)}}, g_{y_{k+1,(1)}}, ..., g_{y_{m,(0)}}, g_{y_{m,(1)}})$ be a satisfiability model for $\Phi'(\tau(\mathbf{z}))$. We now construct a model $M = (f_{y_1}, ..., f_{y_m})$ that satisfies $\Phi(\tau(\mathbf{z}))$. Let $f_{y_i} := g_{y_i}$ for $i = 1, ..., k$, and for $i = k+1, ..., m$, let

$$f_{y_i}(x_{d_{i,1}}, ..., x_{d_{i,n_i}}, x_n) := (x_n \vee g_{y_{i,(0)}}(\mathbf{x_{d_i}})) \wedge (\neg x_n \vee g_{y_{i,(1)}}(\mathbf{x_{d_i}}))$$

such that $f_{y_i}[x_n/0] := f_{y_i}(x_{d_{i,1}}, ..., x_{d_{i,n_i}}, 0) \approx g_{y_{i,(0)}}(x_{d_{i,1}}, ..., x_{d_{i,n_i}})$, and thus:

$$\begin{aligned}&\forall x_1 ... \forall x_{n-1}\ \phi(x_1, ..., x_{n-1}, 0, f_{y_1}, ..., f_{y_k}, f_{y_{k+1}}[x_n/0], ..., f_{y_m}[x_n/0], \tau(\mathbf{z}))\\ \approx\ &\forall x_1 ... \forall x_{n-1}\ \phi(x_1, ..., x_{n-1}, 0, g_{y_1}, ..., g_{y_k}, g_{y_{k+1,(0)}}, ..., g_{y_{m,(0)}}, \tau(\mathbf{z}))\end{aligned}$$

The latter is true, since G is a satisfiability model for $\Phi'(\tau(\mathbf{z}))$. The case $x_n = 1$ is analogous, so $\forall x_1 ... \forall x_{n-1} \forall x_n\ \phi(x_1, ..., x_{n-1}, x_n, f_{y_1}, ..., f_{y_m}, t(\mathbf{z})) = 1$. $\square$

The expressive power of *DQBF** and *QBF** formulas in clausal form depends essentially on the structure of the minimal unsatisfiable subformulas of the bound part of the matrix, so we first recall some well-known properties. A *CNF* formula ϕ is called *minimal unsatisfiable* if and only if ϕ is unsatisfiable and the removal of an arbitrary clause produces a satisfiable formula. A (dependency) quantified Boolean formula $\Phi = Q \bigwedge_{1 \le i \le q} \phi_i$ with *CNF* matrix and without free variables is called *minimal false* if and only if Φ is false and removing an arbitrary clause ϕ_i leads to a true formula. If Φ is purely existentially quantified, it is minimal false if and only if the matrix is minimal unsatisfiable. A clause $L \vee K$ is called an $\exists$-*unit clause* for a formula $\Phi \in$ *DQ2-CNF* if and only if L is a literal over an existentially quantified variable and either $L = K$ or K is a universally quantified literal.

A well-known fact about minimal unsatisfiable propositional *2-CNF* formulas is that they contain at most two unit clauses (see, e.g., [16]). This result can be lifted to minimal false *DQ2-CNF* formulas:

Lemma 2. *(Number of $\exists$-unit clauses)*

1. *A minimal unsatisfiable 2-CNF formula contains at most two unit clauses.*
2. *A minimal false DQ2-CNF formula contains at most two $\exists$-unit clauses.*

Proof. Ad 1: Suppose there is some minimal unsatisfiable formula α with at least three unit clauses, say L_1, L_2 and L_3. Then there are clauses $\neg L_1 \vee P_1^{j_1}, \neg L_2 \vee P_2^{j_2}, \neg L_3 \vee P_3^{j_3}$ for $1 \le j_1 \le t_1, 1 \le j_2 \le t_2, 1 \le j_3 \le t_3$. Please notice that α contains no complementary unit clause $\neg L_i$ and no clauses $L_i \vee K_i$ for some literal K_i. Furthermore, all the literals $P_1^1, ..., P_3^{t_3}$ must be distinct. Let α be such a formula with a minimal number of variables.
After applying unit resolution on the L_i and removing the parent clauses, we obtain a minimal unsatisfiable formula with at least three unit clauses. These are the clauses $P_1^1, ..., P_3^{t_3}$. The variables of L_i do not occur in the resulting formula, which is a contradiction to our initial assumption that α has a minimal number of variables.

Ad 2: Let $\Phi = \forall x_1 ... \forall x_n \exists y_1(\mathbf{x_{d_1}}) ... \exists y_m(\mathbf{x_{d_m}})\ \phi$ be a minimal false formula in *DQ2-CNF* with at least three $\exists$-unit clauses. By expansion of the universal variables, we obtain an existentially quantified formula $\exists \mathbf{y}' \phi' \in \exists$*2-CNF* whose matrix ϕ' is unsatisfiable. A subset of the clauses in ϕ' forms a minimal unsatisfiable formula ϕ''. From the first part of the lemma, we know that ϕ'' contains at most two unit clauses, say L_1 and L_2. These literals are unit clauses in the original formula or come from clauses $U_1 \vee L_1$ or $U_2 \vee L_2$ with universal literals U_1, U_2. That means two $\exists$-unit clauses in ϕ are sufficient to produce two unit clauses L_1 and L_2 in ϕ''. All the other $\exists$-unit clauses in ϕ can be removed without making the formula satisfiable, which contradicts our initial assumption that ϕ is minimal false. □

Subsequently, we assume that all *DQ2-CNF^b* formulas are normalized to have no clauses without an existentially quantified literal. This is justified by the fact that clauses without bound variables can be moved in front of the prefix while preserving the equivalence. And in a 2-clause that contains a universal and a free variable, the universal variable can be omitted. Obviously, clauses consisting only of universal variables are unsatisfiable. We can also assume that there are no clauses ϕ_i without free literals. Otherwise, we could replace such a clause with clauses $\phi_i \vee z$ and $\phi_i \vee \neg z$ for a free variable z that already occurs in the formula. But the transformations also work if we assume $\phi_i^f := 0$ for such clauses without free literals.

The following lemma introduces a handy representation in which the minimal false subsets of the quantified bound parts determine which combinations of the free parts must be true.

Lemma 3. *(MF Skeleton)*
Let $\Phi = Q \bigwedge_{1 \leq i \leq q} (\phi_i^b \vee \phi_i^f)$ be a formula in DQ2-CNF^b with non-empty bound parts ϕ_i^b and free parts ϕ_i^f. Let

$$S(\Phi) := \left\{ \Phi' \mid \Phi' = Q\phi_{i_1}^b \wedge ... \wedge \phi_{i_r}^b \textit{ is minimal false}, 1 \leq i_1, ..., i_r \leq q \right\}$$

be the set of minimal false subformulas of the quantified bound parts of Φ. Then we have the following equivalence:

$$\Phi \approx \bigwedge_{(Q\phi_{i_1}^b \wedge ... \wedge \phi_{i_r}^b) \in S(\Phi)} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f)$$

Proof. Let $M(\Phi) := \bigwedge_{(Q\phi_{i_1}^b \wedge ... \wedge \phi_{i_r}^b) \in S(\Phi)} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f)$ be the right side of the equivalence. From right to left, let $M(\Phi)$ be true for a truth assignment τ to the free variables. Suppose $\tau(\Phi)$ is false. Let $Q\phi' := Q(\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b)$ be the quantified bound parts for which $\tau(\phi_{i_k}^f)$ is false for $1 \leq k \leq r$. Under the assumption that $\tau(\Phi)$ is false, $Q\phi'$ is also false and contains therefore a minimal false subformula, say $Q\phi^* := Q(\phi_{j_1}^b \wedge \ldots \wedge \phi_{j_t}^b)$. Since $\tau(M(\Phi))$ is true, one of the free parts $\phi_{j_1}^f, ..., \phi_{j_t}^f$ must be true for τ. That is a contradiction.
From left to right, let Φ be true for a truth assignment τ to the free variables. Suppose $\tau(M(\Phi))$ is false. Then there is a clause $\phi' := (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f)$ in $M(\Phi)$ for which $\tau(\phi_{i_k}^f)$ is false for $1 \leq k \leq r$. Since $Q(\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b)$ is minimal false, we can conclude that $\tau(\Phi)$ is false in contradiction to our assumption. $\square$

On the basis of Lemmas 2 and 3, we now establish a poly-time transformation from *DQ2-CNF^b* to *∃2-CNF^b*.

Theorem 1. *(DQ2-CNF^b $=_{poly-time}$ ∃2-CNF^b)*
Every DQ2-CNF^b formula Φ can be transformed in time $O(|\forall|^2|\Phi|)$ into an equivalent ∃2-CNF^b formula of length at most $O(|\forall|^2|\Phi|)$, where $|\forall|$ is the number of universal quantifiers in Φ.

Proof. In the following, we treat conjunctions of clauses as sets of clauses. Let $\Phi = Q\{(\phi_i^b \vee \phi_i^f) \mid 1 \leq i \leq q\}$ be a formula in *DQ2-CNF^b* with non-empty bound parts ϕ_i^b and free parts ϕ_i^f. We assume that Φ is forall-reduced, which means each clause contains at most one literal over a universal variable. For universal variables u_1, u_2 (not necessarily distinct), we let $\Phi|u_1, u_2$ denote the formula which contains only those clauses of Φ in which the universal literal is over u_1 or u_2 and those clauses without universals:

$$\Phi|u_1, u_2 := Q\{(\phi_i^b \vee \phi_i^f) \mid \text{every universal literal in } \phi_i^b \text{ is } u_1 \text{ or } u_2, 1 \leq i \leq q\}$$

According to Lemma 3, we have

$$\Phi \approx \bigwedge_{(Q\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b) \in S(\Phi)} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f)$$

where $S(\Phi)$ is the set of minimal false subformulas of the quantified bound parts of Φ. Lemma 2 implies that each minimal false formula in $S(\Phi)$ has at most two $\exists$-unit clauses. In analogy to the above notation, we let $S(\Phi)|u_1, u_2 \subseteq S(\Phi)$ be those minimal false formulas in which every $\exists$-unit clause with a universal literal contains either a literal over u_1 or a literal over u_2. Then the union of $S(\Phi)|u_1, u_2$ for all pairs of universals u_1, u_2 equals $S(\Phi)$:

$$\Phi \approx \bigwedge_{u_1, u_2 \in \forall var(\Phi)} \bigwedge_{(Q\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b) \in S(\Phi)|u_1, u_2} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f)$$

It is not difficult to see that $S(\Phi)|u_1, u_2 = S(\Phi|u_1, u_2)$. Then by applying Lemma 3 backwards, we obtain:

$$\begin{aligned}
\Phi &\approx \bigwedge_{u_1, u_2 \in \forall var(\Phi)} \bigwedge_{(Q\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b) \in S(\Phi)|u_1, u_2} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f) \\
&\approx \bigwedge_{u_1, u_2 \in \forall var(\Phi)} \bigwedge_{(Q\phi_{i_1}^b \wedge \ldots \wedge \phi_{i_r}^b) \in S(\Phi|u_1, u_2)} (\phi_{i_1}^f \vee \ldots \vee \phi_{i_r}^f) \\
&\approx \bigwedge_{u_1, u_2 \in \forall var(\Phi)} \Phi|u_1, u_2
\end{aligned}$$

The prefix of each formula $\Phi|u_1, u_2$ can be simplified, because only u_1 and u_2 occur as universal variables in the matrix, so the other universal quantifiers can be dropped. By universal expansion of u_1, u_2 in $\Phi|u_1, u_2$, we obtain an equivalent existentially quantified formula. Its size is at most four times the length of $\Phi|u_1, u_2$. We perform this expansion for every formula $\Phi|u_1, u_2$ and rename the bound variables, such that different pairs of universal variables u_1, u_2 have distinct bound variables. Now, all existential variables can be moved up front, and the result is an equivalent formula in $\exists$*2-CNF*^{b}. Since there are at most $|\forall|^2$ pairs of $|\forall|$ universal variables, the resulting formula has a length of $O(|\forall|^2|\Phi|)$. $\square$

If the whole formula matrix, including the free variables, is in *2-CNF*, we write *DQ2-CNF** instead of *DQ2-CNF*^{b}. In this special case, the above transformation produces an existentially quantified formula with matrix in *2-HORN*, which can easily be solved in linear time. Together with the costs of the transformation, we would have a complexity of $O(|\forall|^2|\Phi|)$ for determining the satisfiability of a *DQ2-CNF** formula. There is, however, a faster way to solve such formulas without the above transformation. Without loss of generality, we can focus on *DQ2-CNF* formulas without free variables, because a *DQBF** formula with prefix $Q = \forall x_1 ... \forall x_n \exists y_1(x_{d_1}) ... \exists y_m(x_{d_m})$ and free variables $z_1, ..., z_r$ is satisfiable if and only if the formula with prefix $Q' = \forall x_1 ... \forall x_n \exists z_1() ... \exists z_r() \exists y_1(x_{d_1}) ... \exists y_m(x_{d_m})$ and the same matrix is true.

As outlined in [2], a quantified *2-CNF* formula Φ can be represented as a directed graph $G(\Phi)$. The idea is to associate with every clause $L \vee K$ the edges $\neg L \to K$ and $\neg K \to L$ for the nodes $L, \neg L, K$ and $\neg K$. Nodes are called existential or universal if the corresponding variable is existentially or universally quantified. For a unit clause L, we introduce the edge $\neg L \to L$. By computing the strongly connected components of the resulting graph, the satisfiability of the formula can be determined in linear time: it is unsatisfiable if and only if one of the following conditions holds:

1. There is a complementary pair of existential nodes, say y and $\neg y$, in some strongly connected component, which is equivalent to the graph having a path from y to $\neg y$ and a path from $\neg y$ to y.
2. A universal node over x is in the same strong component as an existential node over y, and $\exists y$ precedes $\forall x$ in the prefix of Φ.
3. There exists a path from one universal node to another universal node (possibly both over the same variable).

This idea can also be applied to *DQ2-CNF* formulas. The only necessary modification is to replace condition 2 with the following condition 2': "A universal node over x is in the same strong component as an existential node over y, and y does not depend on x."

For $\Phi = Q\phi \in$ *DQ2-CNF*, notice that if $L_1 \to L_2$ is a path in $G(\Phi)$ then Φ is true if and only if $Q(\phi \wedge (\neg L_1 \vee L_2))$ is true. This can be shown by induction on the path length with the observation that for two clauses $\neg L \vee V$ and $\neg V \vee K$ (both not purely universal) in ϕ, we have $Q\,\phi = 1$ if and only if $Q\,(\phi \wedge (\neg L \vee K)) = 1$, where V may be a universal or an existential literal. Then it is easy to see that each of the conditions implies the unsatisfiability of the given formula.

To show the satisfiability of the formula if none of the above conditions hold, the same marking algorithm as in [2] can be used, with the only modification that we stop for condition 2' instead of 2 if a strong component contains both a universal and an existential node. Then it follows that the marking has the same properties as the one in [2], except that a component containing a universal node over some x contains only existential nodes over variables that depend on x. Then it is clear that we can satisfy the formula in the same way as in the original proof by assigning 0 or 1 to existential variables in purely existential components. The truth value of the other existential variables is derived only from those universals on which they depend, so the quantifier dependencies are respected, and we immediately have the following theorem.

Theorem 2. *DQ2-CNF** *satisfiability is solvable in linear time.*

4 Transformation from $\exists 2$-CNF^{b} to $\exists 2$-$HORN^{b}$

In the following, we consider graphs with the structure from the last section also for $\exists 2$-CNF^b formulas $\Phi = \exists y_1 ... \exists y_m \bigwedge_{1 \leq i \leq q} (\phi_i^b \vee \phi_i^f)$. The idea is to associate only the bound literals with nodes in the graph, whereas the free parts become the labels of the corresponding edges. A clause $L \vee K \vee \phi_i^f$ with bound part $\phi_i^b = L \vee K$ and free part ϕ_i^f is then associated with the labeled edges $\neg L \xrightarrow{\phi_i^f} K$ and $\neg K \xrightarrow{\phi_i^f} L$. A clause $L \vee \phi_i^f$ where the bound part is a unit literal is mapped to an edge $\neg L \xrightarrow{\phi_i^f} L$. Figure 1 (left) shows the graph for the following example:

$$\Phi = \exists a \exists b \ (a \vee b \vee z_1) \wedge (\neg a \vee b \vee z_2) \wedge (a \vee \neg b \vee z_3) \wedge (\neg a \vee \neg b \vee z_4)\,.$$

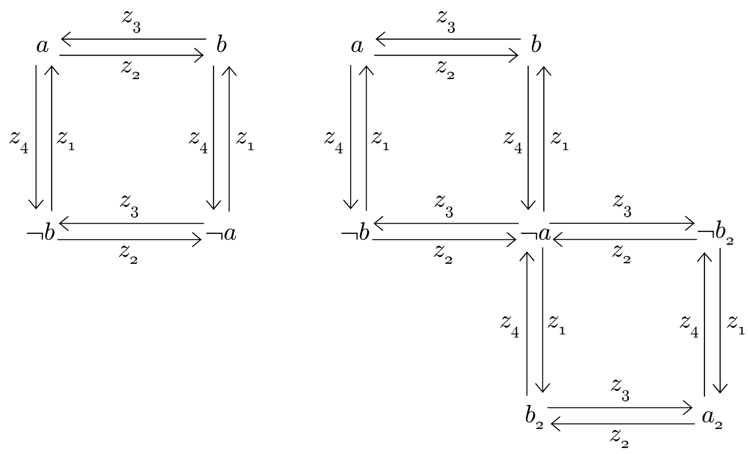

Fig. 1: Example graph (left) and unfolding for variable a (right)

We now translate such graphs into $\exists 2$-$HORN^b$ clauses by mapping an edge $L \xrightarrow{\phi_i^f} K$ to a clause $\neg L \vee K \vee \phi_i^f$. The input formula Φ is satisfiable if and only if there exists an assignment of truth values to the free variables such that for all paths from a node y_k to $\neg y_k$ and back to y_k, at least one edge label evaluates to true. The trick is now to encode this check separately for each quantified variable. That means we create a subformula which is false if and only if there is a path from y_1 to $\neg y_1$ and back to y_1 with all edge labels being false, another subformula for paths from y_2 to $\neg y_2$ and back to y_2, and so on. Furthermore, we unfold the graph for each y_k by "mirroring" it around $\neg y_k$, so that instead of checking for a cycle, it is sufficient to detect a simple path from y_k to $\neg y_k$ and from there to the mirrored copy of y_k. Suitable renamings make sure that all nodes in the unfolded graph have unique names. Figure 1 (right) shows how the graph for the previous example is unfolded for the variable a.

Theorem 3. *($\exists 2$-CNF^b $=_{poly-time}$ $\exists 2$-$HORN^b$)*
Every $\exists 2$-CNF^b formula Φ with $|\exists|$ existential quantifiers can be transformed in time and space $O(|\exists| \cdot |\Phi|)$ into an equivalent $\exists 2$-$HORN^b$ formula.

Proof. Let $\Phi = \exists y_1 ... \exists y_m \bigwedge_{1 \leq i \leq q} (\phi_i^b \vee \phi_i^f) \in \exists 2\text{-}CNF^b$. In addition to the previously stated assumption that the bound parts ϕ_i^b are not empty, we also assume that the quantified bound parts $\exists y_1 ... \exists y_m \bigwedge_{1 \leq i \leq q} \phi_i^b$ yield an unsatisfiable formula. Otherwise, Φ would be true for any truth assignment to the free variables and therefore be a tautology. Furthermore, we do not allow multiple occurrences of identical bound parts. If the formula contains clauses $L \vee K \vee \phi_i^f$ and $L \vee K \vee \phi_j^f$ with the same bound part $L \vee K$, we can replace the first clause with the clauses $L \vee y \vee \phi_i^f$ and $\neg y \vee K \vee \phi_i^f$ for a new existentially quantified variable y.

Let G be the graph associated with Φ as outlined above. The following procedure transforms G into a formula $\Phi^* \in \exists 2\text{-}HORN^b$:

For all bound variables y, compute the graphs $G(y)$ and $G(\neg y)$ by the following renamings with new names $a_y, a_{\neg y}, b_y$:

$G(y)$ is obtained from G by renaming y into a_y and $\neg y$ into $a_{\neg y}$,
all the other nodes are given new unique names.
$G(\neg y)$ is obtained from G by renaming $\neg y$ into $a_{\neg y}$ and y into b_y,
all the other nodes are given new unique names.

For all bound variables y,
compute the combined graph $H(y) := G(y) \cup G(\neg y)$,
with $\mathbf{v}_y$ being the set of names of all nodes in $H(y)$,
build the formula $F(y) := \exists \mathbf{v}_y \; a_y \wedge \neg b_y \wedge \bigwedge_{(L \xrightarrow{\sigma} K) \in H(y)} (\neg L \vee K \vee \sigma)$.

Combine the formulas $F(y_i)$ for the bound variables $y_1, ..., y_m$ in Φ into $\Psi := \exists \mathbf{v}_{y_1} ... \exists \mathbf{v}_{y_m} F(y_1) \wedge ... \wedge F(y_m)$. Clearly, $\Psi \in \exists 2\text{-}HORN^b$.

In order to prove that $\Phi \approx \Psi$, we use the equivalent representations from Lemma 3:

$M(\Phi) := \bigwedge_{(\exists \phi_{i_1}^b \wedge ... \wedge \phi_{i_r}^b) \in S(\Phi)} (\phi_{i_1}^f \vee ... \vee \phi_{i_r}^f) \approx \Phi$

$M(\Psi) := \bigwedge_{(\exists \psi_{j_1}^b \wedge ... \wedge \psi_{j_s}^b) \in S(\Psi)} (\psi_{j_1}^f \vee ... \vee \psi_{j_s}^f) \approx \Psi$

Since the matrix of an existentially quantified minimal false formula is minimal unsatisfiable, we represent the formulas $M(\Phi)$ and $M(\Psi)$ as follows:

$M(\Phi) = \{(\phi_{i_1}^f \vee ... \vee \phi_{i_r}^f) \mid (\phi_{i_1}^b \wedge ... \wedge \phi_{i_r}^b) \text{ minimal unsat, } \phi_{i_k}^b \text{ bound part in } \Phi\}$

$M(\Psi) = \{(\psi_{j_1}^f \vee ... \vee \psi_{j_s}^f) \mid (\psi_{j_1}^b \wedge ... \wedge \psi_{j_s}^b) \text{ minimal unsat, } \psi_{j_l}^b \text{ bound part in } \Psi\}$

Ad $M(\Psi) \models M(\Phi)$: Let $\varphi := \phi_{i_1}^f \vee ... \vee \phi_{i_r}^f$ be a clause in $M(\Phi)$. Then $\beta := \phi_{i_1}^b \wedge ... \wedge \phi_{i_r}^b$ is minimal unsatisfiable, and according to [2], there must be some variable y in β for which a path from y to $\neg y$ and from $\neg y$ to y exists in the graph representing the propositional *2-CNF* formula β. Since the graph G has the same structure, it contains the same path. For fixed y, this path must be unique and have length r, because β would not be minimal unsatisfiable otherwise. Accordingly, there is also exactly one path of length r from a_y to $a_{\neg y}$ and then to b_y in $H(y)$, which implies that the corresponding bound parts of the formula $F(y)$ are minimal unsatisfiable and thus define a clause in $M(\Psi)$. By construction, the path in $H(y)$ is labeled with the same free parts as the corresponding path in G, namely $\{\phi_{i_1}^f, ..., \phi_{i_r}^f\}$. This shows that $M(\Psi)$ contains the clause φ, so $M(\Psi) \models M(\Phi)$.

Ad $M(\Phi) \models M(\Psi)$: This direction is essentially the inverse of the preceding case. Let $\varphi := \psi_{j_1}^f \vee ... \vee \psi_{j_s}^f$ be a clause in $M(\Psi)$. Due to the unique node names, a minimal unsatisfiable subset of bound parts $\psi^b_{j_1} \wedge ... \wedge \psi^b_{j_s}$ in Ψ can only arise within a single formula $F(y)$ for some variable y. The existence of such a minimal unsatisfiable subset of bound parts implies a path of length s from a_y to $a_{\neg y}$ and to b_y in $H(y)$. The path is labeled with $\{\psi^f_{j_1}, ..., \psi^f_{j_s}\}$ and corresponds to a path from y to $\neg y$ and back to y in G with the same edge labels. Such a path implies that there is an unsatisfiable set of bound parts $\phi^b_{i_1} \wedge ... \wedge \phi^b_{i_s}$ in Φ. A subset of these is minimal unsatisfiable, and the corresponding free parts are a subset of the edge labels on the path. It follows that a subset of each clause φ in Ψ is a clause in Φ, and thus $M(\Phi) \models M(\Psi)$. $\square$

5 Conclusion

We have shown that the formula class *DQ2-CNF*^{b} is not significantly more expressive than $\exists$*2-HORN*^{b} and that *DQ2-CNF*^{b} satisfiability is also *NP*-complete. An important intermediate result was a poly-time elimination of all universal quantifiers in a *DQ2-CNF*^{b} or *Q2-CNF*^{b} formula, which might also be useful for *QBF* solvers fighting against the exponential blowup caused by universal expansion in the general case. Along the lines, we have also shown that *DQ2-CNF** satisfiability can be decided in linear time and that universal expansion is also correct for *DQBF**.

While there are formulas for which *DQ2-CNF*^{b} and $\exists$*2-HORN*^{b} are known to be exponentially more concise than propositional *CNF*, the relationship between $\exists$*2-HORN*^{b} and $\exists$*HORN*^{b} remains unclear. The latter class has the same expressive power as Boolean circuits with arbitrary fan-out, which are assumed to be more powerful than propositional formulas. It is not known whether such circuits can also be compactly encoded as poly-size $\exists$*2-HORN*^{b} formulas or, equivalently, whether every $\exists$*HORN*^{b} formula has an equivalent $\exists$*2-HORN*^{b} formula of polynomial length. Perhaps, the poly-time equivalence between (D)*Q2-CNF*^{b} and $\exists$*2-HORN*^{b} can help to shed some light onto this problem. It would also be interesting to investigate whether the transformation between the two classes can be carried out with lower costs than with the procedure that is presented here.

References

[1] S. Aanderaa and E. Börger. *The Horn Complexity of Boolean Functions and Cook's Problem.* Proc. 5th Scandinavian Logic Symposium 1979, Aalborg University Press, pp. 231–256, 1979.

[2] B. Aspvall, M. Plass, and R. Tarjan. *A Linear-Time Algorithm for Testing the Truth of Certain Quantified Boolean Formulas.* Information Processing Letters, 8(3):121–123, 1979.

[3] A. Ayari and D. Basin. *QUBOS: Deciding Quantified Boolean Logic using Propositional Satisfiability Solvers.* Proc. 4th Intl. Conf. on Formal Methods in Computer-Aided Design (FMCAD 2002), Springer LNCS 2517, pp. 187–201, 2002.

[4] L. Babai, L. Fortnow, and C. Lund. *Non-Deterministic Exponential Time has Two-Prover Interactive Protocols.* Journal Computational Complexity, 1(1):3–40, 1991.

[5] A. Biere. *Resolve and Expand.* Proc. 7th Intl. Conf. on Theory and Applications of Satisfiability Testing (SAT 2004), Revised Selected Papers, Springer LNCS 3542, pp. 59–70, 2005.
[6] U. Bubeck and H. Kleine Büning. *Dependency Quantified Horn Formulas: Models and Complexity.* Proc. 9th Intl. Conf. on Theory and Applications of Satisfiability Testing (SAT 2006), Springer LNCS 4121, pp. 198–211, 2006.
[7] U. Bubeck and H. Kleine Büning. *Bounded Universal Expansion for Preprocessing QBF.* Proc. 10th Intl. Conf. on Theory and Applications of Satisfiability Testing (SAT 2007), Springer LNCS 4501, pp. 244–257, 2007.
[8] U. Bubeck and H. Kleine Büning. *A New 3-CNF Transformation by Parallel-Serial Graphs.* Journal Information Processing Letters, 109(7):376–379, 2009.
[9] N. Dershowitz, Z. Hanna, and J. Katz. *Bounded Model Checking with QBF.* Proc. 8th Intl. Conf. on Theory and Applications of Satisfiability Testing (SAT'05). Springer LNCS 3569, pp. 408–414, 2005.
[10] W. Dowling and J. Gallier. *Linear-Time Algorithms for Testing the Satisfiability of Propositional Horn Formulae.* Journal of Logic Programming, 1(3):267–284, 1984.
[11] S. Even, A. Itai, and A. Shamir. *On the Complexity of Timetable and Multi-Commodity Flow Problems.* SIAM Journal on Computing, 5(4):691–703, 1976.
[12] A. Flögel, M. Karpinski, and H. Kleine Büning. *Resolution for Quantified Boolean Formulas.* Information and Computation, 117(1):12–18, 1995.
[13] A. Itai and J. Makowsky. *On the Complexity of Herbrand's Theorem.* Technical Report 243, Technion, Haifa, 1982.
[14] T. Jussila and A. Biere. *Compressing BMC Encodings with QBF.* Electronic Notes in Theoretical Computer Science, 174(3):45–56, 2007.
[15] H. Kleine Büning, X. Zhao, and U. Bubeck. *Resolution and Expressiveness of Subclasses of Quantified Boolean Formulas and Circuits.* Proc. 12th Intl. Conf. on Theory and Applications of Satisfiability Testing (SAT 2009), Springer LNCS 5584, pp. 391–397, 2009.
[16] P. Liberatore. *Redundancy in Logic II: 2-CNF and Horn Propositional Formulae.* Artificial Intelligence, 172(2-3):265–299, 2008.
[17] F. Lonsing and A. Biere. *A Compact Representation for Syntactic Dependencies in QBFs.* Proc. 12th Intl. Conf. on Theory and Applications of Satisfiability Testing (SAT 2009), Springer LNCS 5584, pp. 398–411, 2009.
[18] G. Peterson, J. Reif, and S. Azhar. *Lower Bounds for Multiplayer Non-Cooperative Games of Incomplete Information.* Computers and Mathematics with Applications, 41(7-8):957–992, 2001.
[19] D. Plaisted and S. Greenbaum. *A Structure-preserving Clause Form Translation.* Journal of Symbolic Computation, 2(3):293–304, 1986.
[20] A. Sabharwal, C. Ansotegui, C. Gomes, J. Hart, and B. Selman. *QBF Modeling: Exploiting Player Symmetry for Simplicity and Efficiency.* Proc. SAT 2006, Springer LNCS 4121, pp. 382–395, 2006.
[21] M. Samer and S. Szeider. *Backdoor Sets of Quantified Boolean Formulas.* Proc. 10th Intl. Conf. on Theory and Applications of Satisfiability Testing (SAT 2007), Springer LNCS 4501, pp. 230–243, 2007.
[22] G. Tseitin. *On the Complexity of Derivation in Propositional Calculus.* In A. Silenko (ed.): Studies in Constructive Mathematics and Mathematical Logic, Part II, pp. 115–125, 1970.